Modeling Ellsberg's Paradox in Vague-Vague Cases

Karen M. Kramer **David V. Budescu**
University of Illinois Champaign IL, USA
kkramer@uiuc.edu dbudescu@uiuc.edu

Abstract

We explore a generalization of Ellsberg's paradox (2-color scenario) to the Vague-Vague (V-V) case, in which neither of the probabilities (urns) is specified precisely, but one urn is always more precise than the other. One hundred and seven undergraduate students compared 63 pairs of urns involving positive outcomes. The paradox is as prevalent in the V-V case, as in the standard Precise-Vague (P-V) case. The paradox occurs more often when differences between ranges of vagueness are large and occurs less often with extreme midpoints. The urn with more vagueness was avoided for moderate to high expected probabilities and preferred for low expected probabilities in P-V cases, and the opposite pattern was found for the V-V cases. Models that capture adequately the relationships between the prevalence of vagueness avoidance and the lotteries' parameters (e.g. differences between the two ranges) were fitted for the P-V and V-V cases.

Keywords. Vagueness, ambiguity, imprecise probabilities, Ellsberg's paradox.

1 Introduction

Consider a situation where a Decision-Maker (DM) has to bet on one of two urns containing balls of two colors, say Red and Blue. The composition (proportions of two colors) of one urn is known with certainty, but the composition of the other urn is completely unknown. Imagine that one of the colors (Red or Blue) is arbitrarily made more desirable, simply by associating it with a positive prize of size $x. If DMs are asked to choose one urn when either color is more desirable, many are more likely to select the urn with known content *for both colors.* Ellsberg [10] demonstrated that this choice pattern violates Subjective Expected Utility Theory (SEUT), and this scenario is widely known as the "two-color Ellsberg's paradox".

The most common and appealing explanation of Ellsberg's paradox (e.g., [5]) is that it is due to "ambiguity aversion". Although the term "ambiguity" is often used in this context, we prefer to use the terms "vagueness" or "imprecision" because they describe the situation more accurately (e.g., [4]; [3]). The logic of this explanation is straightforward and compelling ---- *If within each pair, most DMs choose the more precise urn, the modal pattern of joint choices (across the two replications when Red or Blue are the target colors) would, necessarily, lead to the paradox.* Various psychological explanations were offered for the subjects' preference for the more precise urn. Subjects may simply choose the urn about which they have more knowledge and information (Edwards, cited in [24], footnote 4; [1]). The different levels of information may induce various levels of competence [19]. Other, more complex, explanations rely on perception of "hostile nature" [27], anticipation of evaluation by others ([11]; [12]; [13]; [22]; [24]; [25]; [26]), self-evaluation ([11]; [24]; [26]), and others (see reviews by [5] and [8]). Regardless of the underlying psychological reasons, Ellsberg's paradox has become almost synonymous with vagueness avoidance. In fact, most empirical research has focused on single choices between pairs of gambles varying in their precision, and only very few studies (e.g. [23]) have actually replicated the full paradoxical pattern across two choices.

Many researchers have studied and tried to model the behavior underlying this paradox (see [2] and [5] for a comprehensive review; see [6]; [7]; and [9] for typical studies). Most of this research has used Precise-Vague (P-V) cases, where the probabilities of the two colors in one urn are known, but the probabilities in the other urn are vague (specified imprecisely). This work has helped identify some of the factors and conditions that contribute to the intensity of the preference for precision. For example, Einhorn and Hogarth [9] used probability predictions, insurance pricing, and warranty pricing tasks, to show vagueness avoidance at moderate to high probabilities of gains, but vagueness seeking for low probabilities of gains. Kahn & Sarin [21] and Hogarth & Einhorn [20] confirmed these results.

An interesting trend in the literature has been to extend the paradox to new situations. One such generalization was to show that the paradoxical pattern of choices is obtained when the vagueness in the second urn is only partial, i.e., when the DM knows that $Pr(Red) \geq x$, $Pr(Blue) \geq y$, s.t., $0 \leq x,y \leq 1$, but $(x+y) < 1$. This implies that $x \leq Pr(Red) \leq (1-y)$, i.e. Pr(Red) is within a range of size $R=(1-x-y)$ centered at $M=(1+x-y)/2$. Similarly, $y \leq Pr(Blue) \leq (1-x)$, i.e. in a range of size $R=(1-x-y)$ centered at $M=(1+y-x)/2$. The current study follows this trend by extending the paradox to Vague-Vague (V-V) situations where the composition of both urns is only partially specified. Typically, the range of possible probabilities in one urn is smaller than the range of the second urn, but both ranges share the same central value. Thus, $Pr(Red|Urn\ I) \geq x_1$, $Pr(Blue|Urn\ I) \geq y_1$, $Pr(Red|Urn\ II) \geq x_2$, and $Pr(Blue|Urn\ II) \geq y_2$, subject to the constraints: $0 \leq x_1,y_1,x_2,y_2 \leq 1$, $(x_1+y_1) < 1$, $(x_2+y_2) < 1$. Furthermore, $|x_1-y_1| = |x_2-y_2|$, but $R_1=(1-x_1-y_1) \neq R_2=(1-x_2-y_2)$. In other words, $x_1 \leq Pr(Red|Urn\ I) \leq (1-y_1)$ and $x_2 \leq Pr(Red|Urn\ II) \leq (1-y_2)$, and the common midpoint of both intervals is $M=(1+x_1-y_1)=(1+x_2-y_2)$.

The effects of vagueness in P-V cases are relatively well understood (see for example the list of stylized facts in [5]), but the V-V case is more complicated. Becker and Brownson [2] found inconsistencies when they tried to relate vagueness avoidance to differences in the ranges of vague probabilities, and Curley & Yates' studies ([6]; [7]) were inconclusive with regard to the presence and intensity of vagueness avoidance in V-V cases. Curley & Yates [6] examined the choices subjects made in the P-V and V-V case as a function of the width(s) of the interval(s) and the common midpoint of the range of probabilities. They showed subjects were more likely to be vagueness averse as the midpoint increased in P-V cases, but not in V-V cases. Neither vagueness seeking nor avoidance was the predominant behavior for midpoints < .40. The range difference between the two urns was not sufficient for explaining the degree of vagueness avoidance, and no effect of the width of the range was found in preference ratings over the pairs of lotteries.

Undoubtedly, the range difference (wider range – narrower range) is the most salient feature of pairs of gambles with a common midpoint, and one would expect this factor to influence the degree of observed vagueness avoidance [2]. Range difference captures the *relative precision* of the two urns, and DMs who are vagueness averse are expected to choose the more precise urn more often. In fact, it is sensible to predict a positive monotonic relationship between the relative precision of a pair of urns and the intensity of vagueness avoidance displayed by DMs. It is surprising that Curley and Yates could not confirm this expectation, and we plan to study this case in more detail in the current study.

However, the relative precision of a given pair can not fully explain the DM's preferences in the V-V case. Consider, for example, the following three urns: Urn A: $0.45 \leq p \leq 0.55$; Urn B: $0.30 \leq p \leq 0.70$; Urn C: $0.15 \leq p \leq 0.85$, where p is the probability of the desirable event (Red or Blue ball). All urns have a common midpoint (0.5) but vary in their (im)precision. Urn A has a range of 0.10, Urn B has a range of 0.40, and Urn C spans a range of 0.70. Imagine that a DM has to choose between A and B, and between B and C. In both pairs the range difference (relative precision) is the same (0.30), but vagueness avoidance is expected to be stronger for the A,B pair, because most people would prefer the higher certainty associated with A. If, on the other hand, there is a fair amount of vagueness in both urns, people may feel that vagueness is unavoidable, and may focus their attention on other features. For example, they may notice that, in the best possible case, Urn C offers a very high probability (0.85). This shift of attention may reduce the tendency to avoid vagueness and may lead to indifference or vagueness seeking.

This example highlights the importance of the more precise urn. The range of probabilities in this urn represents the highest possible (an upper bound on) precision, and we refer to it as the pair's *absolute imprecision*. We predict that, everything else being equal, vagueness avoidance should increase as the absolute imprecision decreases. Conversely, as absolute imprecision increases (i.e., as the more precise urn becomes more vague), we should observe more instances of indifference between the two urns, and an increased tendency of vagueness preference.

2 The Current Study

The purpose of the present study is to examine DMs' choices in the presence of vagueness and their tendency to succumb to Ellsberg's paradox, and to test a variety of models for these choices. We will be especially concerned with the V-V case, where both lotteries are imprecise and will contrast them with the choices in the "standard" P-V case. The experiment will focus on V-V choices in the domain of gains, using a design similar to the one employed by Curley & Yates [6]. We will, however use a much larger number of V-V pairs covering more ranges at three different midpoints. The subjects' choices in each pair will be classified as vagueness seeking, vagueness avoiding or indifferent to vagueness, and the proportions of vagueness avoidance choices will be analyzed as a function of the lower and upper bounds (that can be converted to measures of absolute imprecision and relative precision) and the midpoint of the pair. As indicated earlier, vagueness avoidance is expected to increase with relative precision (the difference between the two intervals) and with decreases in absolute imprecision (the width of the narrower interval).

There is empirical evidence that the intensity of vagueness avoidance increases with midpoint ([6]; [9]), and the midpoint may interact with the two precision measures of a pair. We expect pairs with low midpoints and high relative precision will induce less vagueness avoidance than pairs with high midpoints and high relative precision. If the more precise urn's range is not much different from the other urn's range, people are expected to feel more indifferent (and possibly be more vagueness seeking) between the urns.

Thus, the experiment will allow us to verify the presence of the paradoxical pattern in the V-V case, document its prevalence and compare it with the P-V case. The prevalence of the paradox will be analyzed as a function of the midpoint, range widths, and/or range differences. In general, we expect that those factors that induce higher levels of vagueness avoidance will also increase the frequency of the paradoxical pattern, but an intriguing question that was never fully examined is whether the occurrence of the paradox can be predicted precisely from the subjects' attitudes towards precision. We expect Ellsberg's paradox to be the modal, but not the universal, pattern. In those cases when the paradox does not occur, we predict different patterns as a function of the common midpoint. We expect subjects to exhibit more indifference for pairs with a midpoint of 0.50, where it is easier, and more natural, to either imagine symmetric distributions of probabilities ([11]; footnote 8), and/or a greater number of possible distributions [10], than with extreme midpoints. On the other hand, we expect subjects to be consistent with SEUT more often with extreme midpoints, where the imagined distributions are more likely to be asymmetric and to be skewed in opposite directions.

3 Method

Subjects 107 undergraduates in an introductory psychology class at the University of Illinois. They received an hour of credit, and had a chance to win additional money at the end of the experiment.

Stimuli. The stimuli consisted of 63 different pairs of urns, with 100 marbles in each urn. The colors of marbles in the two urns were red and blue. The pairs varied in terms of the (common) midpoint, and the ranges of values in each urn. Fifteen pairs had a midpoint of 20, fifteen pairs had a midpoint of 80, and thirty-three pairs had a midpoint of 50. The midpoint is equivalent to the "expected" number of red marbles in each urn (under a uniform distribution). Six different ranges were used with a midpoint of 20 or 80, and ten ranges were used with a midpoint of 50. Ranges included 0 (precise probabilities) and 100 (complete vagueness).

Procedure. Subjects were run individually on personal computers in a lab. Each of the 63 pairs was presented twice. One presentation implied a desirable outcome was associated with the acquisition of a red marble. In the other presentation, the desirable outcome was associated with the acquisition of a blue marble. These 126 pairs were presented one at a time in a different randomized order for each subject. For each pair, a subject decided whether to select Urn I, Urn II, or either urn (i.e. express indifference), but we made an explicit effort to dissuade subjects from selecting "either urn", by emphasizing that in such cases their choice would be determined by random device.

In one group (80 subjects) the urn with the narrower range was always presented on the left; in the second group (27 subjects) the placement of the urn with a narrower range (left or right) was randomly determined on every trial. We did not find any significant differences between the two conditions, so the data from the two groups were combined. The average subject completed both parts of the experiment in approximately 30 minutes. At the conclusion of the experiment, a pair of urns was chosen, and the subjects' choices for each color were noted. If the color of the selected marble matched the target color, the subject won \$3. Otherwise, the subject did not receive any money. Twenty-one subjects received \$0, 59 gained \$3, and 27 gained \$6.

4 Results

4.1 Analysis of joint choice patterns

Distribution of Responses. For any given pair of urns there are nine distinct patterns of responses, that can be classified into five general patterns: CP -- Classic Paradox -- the DM selects the more precise urn twice; RP -- Reverse Paradox -- The DM selects the less precise urn twice; I -- Indifference --- The DM expresses indifference between the two bets twice; C -- Consistency -- The Dm chooses to bet on the more precise urn once, and on the more vague urn once; and WI -- Weak Indifference -- The DM expresses a preference for one urn on one occasion and indifference on the other. Clearly, indifference and consistency comply with SEUT while weak indifference does not allow an unequivocal test of the paradox.

The distribution of the nine choice patterns are summarized in Tables 1 and 2 (for P-V and V-V, respectively). In each table, panels 1–3 contain information for each midpoint separately (we present only a subset of the results for the midpoint of 50, namely the choices in those ranges that were used for all midpoints) and panel 4 is a summary across all midpoints based on the subset of common ranges. The marginal distributions indicate the predominance of vagueness avoiding for each color and each midpoint, for P-V and V-V alike. They also indicate a higher percentage of vagueness seeking than indifference for extreme midpoints (20 and 80), and a reversed trend (more indifference than vagueness seeking) for a midpoint of 50.

Midpoint = 20	Blue			
Red (n=535)	VA	I	VS	Total
VA	33.60	3.70	7.10	44.40
I	9.20	8.60	1.90	19.70
VS	24.90	2.80	8.20	35.90
Total	67.70	15.10	17.20	100.00
Midpoint=50	**Blue**			
Red (n=535)	VA	I	VS	Total
VA	38.70	6.00	6.90	51.60
I	6.70	17.80	3.00	27.50
VS	7.20	3.00	10.70	20.90
Total	52.60	26.80	20.60	100.00
Midpoint = 80	**Blue**			
Red (n=535)	VA	I	VS	Total
VA	32.70	7.30	20.00	60.00
I	5.80	9.20	2.20	17.20
VS	11.70	2.10	9.00	22.80
Total	50.20	18.60	31.20	100.00
All Midpoints	**Blue**			
Red (n=1605)	VA	I	VS	Total
VA	35.00	5.70	11.30	52.00
I	7.20	11.80	2.40	21.40
VS	14.70	2.60	9.30	26.60
Total	56.90	20.10	23.00	100.00

Table 1. Percentage of each observed pattern for the P-V case (VA = vagueness avoidance, I = indifference, VS = vagueness seeking)

For both cases, and all midpoints, the classic paradox was the modal choice. Consistency was a close second, and the reverse paradox was the most rare pattern. In general, the results for P-V and V-V pairs were similar, but there were some interesting differences. Consistent subjects could chose the less vague urn for a red marble, and the more vague urn for a blue marble, or they could chose the less vague urn for blue and the more vague urn for red. For the P-V case, subjects chose the more vague urn for the low midpoint, and the less vague urn for the high midpoint. An opposite trend was observed for V-V cases. Thus, as expected, vagueness avoidance increased with midpoint (per color) for P-V cases and decreased with midpoint for V-V cases. In addition, indifference was almost twice as prevalent for a midpoint of 50 than for the other two midpoints. Conversely, consistency was twice as frequent for extreme midpoints than for the midpoint of 50.

The marginal frequencies of the tables document the predominance of vagueness avoidance and the upper left cell indicates that the classic paradox is the modal pattern. A natural question is whether the frequency of the paradox can be predicted exclusively from the subjects' global tendency to choose the more precise lottery. In other words, is Pr(Classic paradox)=Pr(VA|Red) x Pr(VA|Blue)? Somewhat surprisingly, the answer is negative: In all tables the paradox occurred more frequently than one would predict from independent vagueness avoidance choices (overall, 5.83% above expectation). Similarly, the indifferent pattern and the reverse paradox were under-predicted by the marginal distributions (by 7.67% and 3.60%, respectively), and all the off-diagonal cells were over-predicted. This indicates that the rate of the various patterns (e.g. CP) is not driven exclusively by a constant tendency to avoid/prefer vagueness, and it is determined, at least in part, by the parameters of the specific pairs being compared.

Midpoint = 20	Blue			
Red (n=1070)	VA	I	VS	Total
VA	28.50	6.20	18.30	53.00
I	6.20	8.20	4.60	19.00
VS	13.40	2.10	12.50	28.00
Total	48.10	16.50	35.40	100.00
Midpoint=50	**Blue**			
Red (n=1070)	VA	I	VS	Total
VA	33.40	6.80	7.00	47.20
I	5.50	20.90	3.70	30.10
VS	7.10	4.40	11.10	22.60
Total	46.00	32.10	21.80	100.00
Midpoint = 80	**Blue**			
Red (n=1070)	VA	I	VS	Total
VA	30.50	4.70	13.40	48.60
I	6.90	8.70	2.50	18.10
VS	18.80	3.20	11.40	33.40
Total	56.20	16.60	27.30	100.00
All Midpoints	**Blue**			
Red (n=3210)	VA	I	VS	Total
VA	30.80	5.90	12.90	49.60
I	6.20	12.60	3.60	22.40
VS	13.10	3.20	11.70	28.00
Total	50.10	21.70	28.20	100.00

Table 2. Percentages of each observed pattern for the V-V case. (VA = vagueness avoidance, I = indifference, VS = vagueness seeking)

Log-Linear Models. The frequency of each of the five patterns was tabulated as a function of the urns' midpoint and difference between the two ranges (wider range - narrower range). Log-linear models were fit to each pattern, to determine the effect of the two factors on the observed frequency of the target pattern. The saturated model is:

$$ln(f_{ij}) = l + l_{R(i)} + l_{C(j)} + l_{RC(ij)} \qquad (1)$$

where R is the row (midpoint) effect, C is the column (range difference) effect, and RC is the interaction of these effects. Reduced models are defined by constraining some of the parameters to equal 0. The fit of reduced versions of model (1) is presented in Table 3.

The P-V Case	Range Difference				
Midpoint	2	20	30	38	40
20	32	25	42	41	40
50	27	39	47	47	47
80	27	32	41	36	39

Classic Paradox in the P-V case (N=562)			*(if ≈1, model fits)
Model	df	G^2	G^2/df
Complete Independence	8	3.37	.42 *
Midpoint Only	12	18.00	1.50
Range Diff. Only	10	6.49	.65 *

The V-V Case	Range Difference								
Mid point	2	8	10	18	20	28	30	36	38
20	20	27	57	62	35	42	--	27	35
50	47	21	58	112	84	47	79	46	48
80	17	18	50	75	30	49	--	41	46

Classic Paradox in the V-V case (N=988)			*(if ≈1, model fits)
Model	df	G^2	G^2/df
Complete Independence	14	12.33	.88 *
Midpoint Only	21	178.61	8.51
Range Diff. Only	16	16.47	1.03 *

Table 3. Log linear analysis of frequency of the Classic Paradox (CP)

For each model we report the degrees of freedom (df), the likelihood ratio (G^2) and the ratio G^2/df. Usually, the models' goodness of fit is tested by comparing G^2 with its asymptotic sampling distribution (χ^2). In this case, however, this would be inappropriate because the observations are not independent. Instead, we use the ratio G^2/df as a heuristic measure of the fit of a model. In general, the closer the G^2/df ratio is to 1, the better the fit of the model (e.g., [16], [17]; [18], ch. 17).

In both cases, the reduced model including the range difference effect alone appeared best. The midpoint alone was not as influential. Thus, it appears that range difference (relative precision) was the most important predictor of the incidence of CP.

4.2 Analysis of choices within a single gamble

Distribution of responses. In most cases subjects tend to choose the more precise of the two gambles in a pair. Across the 9,630 cases examined, the more precise option was chosen in 51.36%, vagueness preference was observed in 26.99% of the cases, and subjects expressed indifference towards (im)precision on 21.65% of the occasions. This general pattern held for extreme midpoints, for both colors and for the two types of pairs (P-V and V-V). The distribution over the three choices varied slightly over midpoints, colors, and types of pairs (in particular, for the midpoint of 50 indifference was more prevalent than vagueness preference). However, the distinct preference for precision was almost constant across all cases. This pattern held for most individual subjects as well: 84 out of 107=79% for P-V, and 83 out of 107=78% for V-V displayed vagueness avoidance much more frequently than vagueness seeking.

Logit models of vagueness avoidance. We modeled vagueness avoidance as a function of the common midpoint, and the levels of vagueness (widths of the ranges) of the two gambles in each pair. The response variable was the logit of the probability of selecting the more precise urn, i.e., log{number of vagueness avoidance choices/ number of vagueness seeking choices}, excluding indifferent responses. The logit models assume that the subject weighs differentially the upper and lower bounds of any range of probabilities, and these weights indicate which of the bounds is more salient [3]. This formulation is very similar to Ellsberg's model ([10], p. 665), who speculated that the DMs take a weighted average of the lower bound and the midpoint of the range. However, the midpoint is simply the average of the lower and upper bound, so his model can be mapped into ours by a simple linear transformation.
Let the “subjective probability equivalent ” v_i of urn i (i=1,2) be:

$$v_i = w\, \ell_i + (1-w)\, u_i \,, \qquad \text{s.t. } 0 \le w \le 1, \qquad (2)$$

where ℓ_1 and ℓ_2 are the lower limits, and u_1 and u_2 are the upper limits, of the narrower and wider range, respectively. The probability of choosing the urn with the narrower range (v_1) over the one with the wider (v_2) range is modeled by:

$$p_{12} = 1 / [1 + \exp\{v_1 - v_2\}] \,, \qquad (3)$$

or, alternatively, by:

$$\ln[p_{12} / (1 - p_{12})] = (v_1 - v_2) = w\,(\ell_1 - \ell_2) + (1-w)(u_1 - u_2). \qquad (4)$$

The estimated parameter can help us determine whether subjects put greater weight on the lower or the upper bounds of the vagueness ranges. If w<.5, subjects weight the upper limit of a range more and are vagueness seeking. If w>.5, subjects weight the lower limit of a range more and are vagueness avoiding. If w=0.5, subjects are insensitive to the vagueness of the range. This model uses a single weight, w, for both urns implying that the choice between them depends solely on the difference between the widths of the intervals, to which we referred earlier as relative precision.

An alternative, more general, model allows for the possibility that subjects focus on various bounds for the two urns (say the lower bound of narrower urn, and the upper bound of more vague urn). The "subjective probability equivalent" of urn i (i=1,2), v_i, is:

$v_i = w_i \ell_i + (1-w_i) u_i$, s.t. $0 \le w_i \le 1$ (i=1, 2), (5)

where, w_1 and w_2 are the weights for the narrower and the wider range, respectively. The probability of choosing the narrower range over the wider range is modeled by:

$\ln[p_{12} / (1- p_{12})] = w_1(\ell_1-u_l) - w_2 (\ell_2-u_2) + (u_1-u_2)$ (6)

The interpretation of each w is the same as in model 1. It is also meaningful to compare the two weights. For example, if $|w_1-.5| > |w_2-.5|$, the salience of the narrower range is greater than that of the wider range.

The P-V case

Mid Point	G^2/df (df)	adj. R^2	Effect	w	se(w)
20	4.092 (4)	.751	Range difference*	.510	.003
50	7.921 (8)	.750	Range difference*	.516	.001
80	9.151 (4)	.750	Range difference*	.537	.003
All	10.545 (18)	.750	Range difference*	.518	.001
All	1.599 (17)	.750	Midpoint* Range difference*	- .505	- .002

The V-V case

Mid Point	G^2/df (df)	adj. R^2	Effect	w	se(w)
20	10.931 (9)	.750	Range difference*	.525	.003
50	5.562 (23)	.750	Range difference*	.517	.001
80	6.700 (9)	.751	Range difference*	.521	.002
All	6.940 (43)	.750	Range difference*	.519	.001
All	6.216 (42)	.750	Midpoint* range difference*	- .514	- .001

Table 4. Logit model 1 for all midpoints (*: $p < 0.05$)

Both models were fit for each midpoint separately, and across all midpoints (by allowing another parameter for the midpoint). The results for the first model (Eq 2) are summarized in Table 4. The models fit the data exceptionally well (maximum G^2/df = 10.931 and minimum R^2_{adj} = .75). The model for midpoint 20, V-V cases, had the worst fit. For all midpoints and for the total combined sample in both sets of cases, range difference (relative precision) predicted quite well the probability of choosing the narrower width urn (it was significant in all cases). In the combined sample the midpoint effect was also significant indicating that the degree of vagueness aversion varied across the three midpoints. Surprisingly, the midpoint effect had a slightly positive relationship with the degree of vagueness aversion in V-V cases, which counters our earlier findings that vagueness avoidance seemed to *decrease* with midpoint in these cases. But this effect was not as strong as the positive relationship between midpoint and vagueness avoidance for the P-V case. Most relevant to our purposes, all the estimated w values were close to, but systematically greater than, 0.5. Therefore, the model found a slight, yet significant, degree of vagueness aversion. Finally, note this tendency was somewhat stronger for the V-V case.

Mid Point	G^2/df (df)	adj. R^2	Effect	se(w_i)	w_i
20	2.416 (8)	.750	Narrow range Wide range*	.004 .002	.493 .517
50	3.197 (22)	.750	Narrow range* Wide range*	.002 .001	.507 .515
80	5.196 (8)	.750	Narrow range* Wide range*	.004 .002	.536 .525
All	5.497 (42)	.750	Narrow range* Wide range*	.001 .001	.508 .516
All	5.555 (41)	.750	Midpoint Narrow range* Wide range*	.001 .001 .001	-- .508 .515

Table 5. Logit model 2 for all midpoints for the V-V case (*: $p < 0.05$)

The second model (Eq 5) was fitted to the V-V cases (the model can not fit the P-V items because the lower and upper bounds are identical), and the results are displayed in Table 5. Evidently, these models had equivalent, or slightly better, fit than the corresponding first set of models (maximum G^2/df = 5.555). The worst fit was found across midpoints. The effects of the narrower *and* the wider range were significant in all cases, except for the narrower range at a midpoint of 20. The midpoint effects was not significant. For middle and high midpoints, the larger the wide range and/or the smaller the narrow range, the more likely a subject was to be vagueness avoiding. For the low midpoint, the increase in absolute imprecision was associated with higher levels of vagueness avoidance. All w_i values were slightly greater than .5. Interestingly, there was a slight increase in the values of both w_i across midpoints, suggesting that subjects were slightly more vagueness avoiding for higher midpoints.

As hypothesized, increased relative precision was associated with higher levels of vagueness avoidance. Also, as predicted, increased absolute imprecision (a wider narrower range) led to a decrease in vagueness avoidance. This latter pattern was obtained for the middle and high midpoints, but the opposite was true for the low midpoint.

5 Summary and Discussion

This study shows that, in most cases, a majority of people prefer precisely specified gambles and succumb to Ellsberg's paradox in "dual vagueness" (V-V) cases. The tendency to avoid the more vague urn and the prevalence of the classic paradox is highly similar in the P-V and the V-V situations with some subtle, but systematic, differences. In this section we summarize the major findings and discuss several reasonable explanations. These are not the only feasible accounts of the process underlying the observed choices, and we cannot categorically reject alternative explanations based on the available evidence. In fact, having documented the presence of the paradox in the V-V case, we think that the major challenge for researchers in this area is to identify the best fitting and most parsimonious model.

The prevalence of the paradoxical pattern of choices depends primarily on the ranges of the two gambles (i.e., the pair's relative, and absolute precision) and, to a lesser degree, on the pair's common midpoint. The differences between the P-V and V-V cases are best understood if we consider the results for the various midpoints separately. Consider first the differences observed between the two extreme midpoints. For the P-V case there is less vagueness avoidance (and more vagueness seeking) for the low midpoint (20), than for the high midpoint (80). On the other hand, for V-V pairs, we found more vagueness avoidance (and less vagueness seeking) for the low midpoint than for the high midpoint. This difference is clearly reflected in the results for the two consistent patterns: Although the overall level of consistency is about equal for the two types, we find that the type of consistency varies as a function of the midpoint and the nature of the pair! As the midpoint increases, there is a higher tendency to choose the more precise gamble in a P-V pair, whereas in the V-V case vagueness avoidance decreases with increasing midpoint (see similar results in [6]; [9]; and [14]; [15]).

What accounts for the difference between the two types? In the P-V case the precise urn provides a clear reference point and the DMs can focus, almost exclusively, on the features of the vague urn. Its upper limit offers an attractive probability (higher than the precise), but this is accompanied with the risk of a lower probability (towards the lower limit). The subjects' behavior in these cases indicates that when the precise probability is "sufficiently high" (i.e., high midpoint) they resist the temptation of the upper limit and prefer the security of the precise urn (hence, the high level of vagueness avoidance). But for low midpoints the security offered by the precise option is not sufficient, and there is a higher tendency to opt for the vague urn, presumably because of its attractive upper limit.

We can use these limits to describe subjects' behavior in V-V cases as well. Here there is no "guaranteed security level" and one would expect the DM to focus on the lower limits to ascertain the guaranteed security level in each urn. The higher security level can always be found in the more precise urn, hence for low midpoints DMs are likely to choose the more secure (i.e., the more precise) urn. This is the pattern captured by the parameters' estimates in all the models. However, the concern with security should decrease for higher midpoints. Indeed, we see that vagueness avoidance decreases as a function of the midpoint of the urns.

An alternative explanation for behavior with V-V cases is that when comparing two vague urns, subjects focus on the information available about the frequency of the two colors and imagine that the unknown balls in the urn are distributed according to the same rule. Consider two hypothetical urns with the same (high) midpoint of 70 Red balls. If the DM knows that in Urn A there are 50 Red balls and 10 Blue balls (so, the number of Reds is between 50 and 90), he/she may guess that the ratio of Red and Blue among the other (unknown) 40 balls is also 5:1. The DM's best guess would be that (100*5/6=) 83 of the balls in Urn A are Red and (100-83=) 17 are Blue. Imagine that in Urn B there are 60 Red balls and 20 Blue (so the number of Reds is between 60 and 80). The DM may infer that the ratio of the two colors is the same for the 20 unknown balls, and his/her best guess would be that (100*3/4=) 75 of the balls in Urn B are Red, and the remaining (100-75=) 25 are Blue. If the prize is associated with the drawing of a Red ball, the DM is more likely to choose the more vague Urn A, because he/she would expect it to have more Red balls. If however the DM had to choose between the two urns when Blue balls are desirable (low midpoint = 30), he/she would be more likely to pick the more precise Urn B. This is, indeed, the observed pattern in the data.

Another observed regularity is that there are more consistent choices for extreme midpoints, and a higher rate of indifference for the central value of 50. This can be attributed to the symmetry that underlies all the decisions for the 0.50 midpoint (P-V and V-V alike). Most, if not all, hypothetical and imagined distributions over the range are symmetric and the midpoint is the most salient focal point of the range, regardless of the interval width. This increases the likelihood of indifference between the two urns. For the extreme midpoints the most salient feature is the asymmetry between the two colors, and this feature favors consistent choices over indifference. Note that the two scenarios that were described in the previous paragraphs (for the P-V and V-V cases, respectively) predict opposite choices for the low and high midpoints. And, of course, opposite choices for the two extremes, imply consistency!

Becker & Brownson [2] were the first to suggest that subjects are sensitive to the amount of information is in each pair of urns when making their decisions. The log-linear models confirmed the relevance of the relative precision as a predictor of *the rate of paradoxical pattern*, and the logit models results confirmed its importance for predicting the *rate of vagueness avoidance within single pairs*. These results indicate, unequivocally, that as relative precision increases vagueness avoidance (and the tendency to succumb to the famous paradox) increases. Interestingly, this contradicts one of the conclusions drawn by Curley & Yates [6] who determined that "ambiguity avoidance did not significantly increase with the interval range". Relative precision is not the single predictor of the regularities in the data. We have argued that its effects are contingent on the absolute imprecision in a pair, as measured by the width of the narrower interval. This prediction was also confirmed by our analyses.

The logit models allowed us to estimate a parameter, w, that captures the relative salience of the intervals' endpoints. This parameter can be interpreted as a function of its location relative to the bounds (w=0 or 1), and the point of indifference (w=0.5). Indeed, all logit models that were fitted found $w \neq 0.5$, indicating subjects weigh the limits of both urns differently. The simpler model, which only relies on relative precision reveals that in all the cases $w > 0.5$, consistent with systematic vagueness avoidance. The over-weighting of the lower limits of the urns is slightly more pronounced for higher midpoints in P-V cases. The second model distinguishes between the two ranges by allowing the fit of different weights and this improves the model's fit, but the improvement is quite modest. Again, all the weights point to vagueness avoidance.

Most subjects displayed vagueness avoidance more often than any other behavior, and the parameters estimated by the two models captured this tendency, at least in a qualitative sense. We were, however, surprised and somewhat disappointed that the estimated weights were so close to the neutral point of indifference, 0.5. Simply put, their magnitude does not seem to match the strong and obvious vagueness avoidance observed. It is not clear at this point whether this is a weakness of this particular form of the model, or an artifact of the experimental design (allowing to many indifferences and relying too much on results aggregated over many subjects), or the fitting procedure employed. We hope that further research would shed more light on these possibilities.

Acknowledgments

This work was supported in part by NSF grant SBR 96-32448.

References

[1] Baron, J., & Frisch, D. (1994). Ambiguous probabilities and the paradoxes of expected utility. In G. Wright & P. Ayton (eds.) *Subjective Probability*. John Wiley & Sons Ltd.

[2] Becker, S.W., & Brownson, F.O. (1964). What price ambiguity? Or the role of ambiguity in decision-making. *Journal of Political Economy*, 72, 62-73.

[3] Budescu, D.V., Kuhn, K.M., & Kramer, K.M. (1998). *Beyond Ellsberg's Paradox: Modeling the Effects of Vagueness in Risky Decisions*. Paper presented at the 4th Annual French Conference on Experimental Economics, Cachan, France.

[4] Budescu, D. V., Weinberg, S., & Wallsten, T. S. (1988). Decisions based on numerically and verbally expressed uncertainties. *Journal of Experimental Psychology: Human Perception and Performance*, 14, 281-294.

[5] Camerer, C., & Weber, M. (1992). Recent developments in modeling preferences: Uncertainty and ambiguity. *Journal of Risk and Uncertainty*, 5, 325-370.

[6] Curley, S. P., & Yates, J. F. (1985). The center and range of the probability interval as factors affecting ambiguity preferences. *Organizational Behavior and Human Decision Processes*, 36, 273-287.

[7] Curley, S. P. & Yates, J. F. (1989). An empirical evaluation of descriptive models of ambiguity reactions in choice situations. *Journal of Mathematical Psychology*, 33, 397-427.

[8] Curley, S. P., Yates, F. J., & Abrams, R.A. (1986). Psychological sources of ambiguity avoidance. *Organizational Behavior and Human Decision Processes*, 38, 230-256.

[9] Einhorn, H.J., & Hogarth, R.M. (1986). Decision making under ambiguity. *Journal of Business*, 59, S225-S250.

[10] Ellsberg, D. (1961). Risk, ambiguity, and the Savage axioms. *Quarterly Journal of Economics*, 75, 643-669.

[11] Ellsberg, D. (1963). Risk, ambiguity, and the Savage axioms: Reply. *Quarterly Journal of Economics*, 77, 336-342.

[12] Fellner, W. (1961). Distortion of subjective probabilities as a reaction to uncertainty. *Quarterly Journal of Economics*, 75, 670-689.

[13] Gardenfors, P. (1979). Forecasts, decisions, and uncertain probabilities. *Erkenntis*, 14, 159-181.

[14] Gardenfors, P., & Sahlin, N.E. (1982). Unreliable probabilities, risk taking, and decision making. *Synthese*, 53, 361-386.

[15] Gardenfors, P., & Sahlin, N.E. (1983). Decision making with unreliable probabilities. *British Journal of Mathematical and Statistical Psychology*, 36, 240-251.

[16] Goodman, L.A. (1971). The analysis of multidimensional contingency tables: stepwise procedures and direct estimation methods for building models for multiple classifications. *Technometrics*, 13, 33-61.

[17] Goodman, L.A. (1975). On the relationship between two statistics pertaining to tests of three-factor interaction in contingency tables. *Journal of the American Statistical Association*, 70, 624-625.

[18] Haberman, S.J. (1978). *Analysis of Qualitative Data*, New York: Academic Press.

[19] Heath, C., & Tversky, A. (1991). Preference and belief: Ambiguity and competence in choice under uncertainty. *Journal of Risk and Uncertainty*, 4, 5-28.

[20] Hogarth, R.M., & Einhorn, H.J. (1990). Venture theory: A model of decision weights. *Management Science*, 36, 780-803.

[21] Kahn, B.E., & Sarin, R.K. (1988). Modeling ambiguity in decisions under uncertainty. *Journal of Consumer Research*, 15, 265-272.

[22] MacCrimmon, K.R. (1968). Descriptive and normative implications of the decision theory postulates. In K. Borch and J. Mossin (eds.), *Risk and Uncertainty*. London: MacMillan.

[23] MacCrimmon, K.R., & Larsson, S.(1979). Utility theory: Axioms versus "paradoxes." In Allais and O. Hagen (eds.), *Expected Utility and the Allais Paradox*. Dordrecht, Holland: D. Reidel, 333-409.

[24] Roberts, H.V. (1963). Risk, ambiguity, and the Savage axioms: Comment. *Quarterly Journal of Economics*, 77, 327-336.

[25] Slovic, P., & Tversky, A. (1974). Who accepts Savage's axiom? *Behavioral Science*, 19, 368-373.

[26] Toda, M., & Shuford, E. H., Jr. (1965). Utility, induced utilities, and small worlds. *Behavioral Science*, 10, 238-254.

[27] Yates, J. F. & Zukowski, L. G. (1976). Characterization of ambiguity in decision making. *Behavioral Science*, 21, 19- 25.