Consumption/Pollution Tradeoffs under Hard Uncertainty and Irreversibility*

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Abstract

This paper deals with a model of pollution accumulation in which a catastrophic environmental event occurs once the pollution stock exceeds some uncertain critical level. This problem is studied in a context of "hard uncertainty" since we consider that the available knowledge concerning the value taken by the critical pollution threshold contains both randomness and imprecision. Such a general form of knowledge is modelled as a (closed) random interval. This approach is mathematically tractable and amenable to numerical simulations. In this framework we investigate the effect of hard uncertainty on the optimal pollution/consumption trade-off and we compare the results with those obtained both in the certainty case and in the case of "soft uncertainty" (where only randomness prevails).

Keywords. optimal pollution control, environmental risk, belief functions, random intervals, representation of uncertainty.

1 Introduction

This paper presents an optimal pollution control model in which individuals have to tradeoff consumption against pollution given that consumption gives rise to pollution. We develop a partial equilibrium model for a polluting economy since we neither consider the capital accumulation nor the production process for the final good. The consumption decisions are made under uncertainty. Indeed, when the pollution stock exceeds some unknown critical threshold a catastrophic environmental event occurs. Such an environmental catastrophe is irreversible in the sense that once it has occurred the economy can not recover its initial state. Similar situations have been studied among others by Cropper [7], Conrad [6], Pethig [12] and more recently by Clarke and Reed [5] and Tsur and Zemel [13], [14] and [15].

The main difference between these works and our lies in the form of uncertainty we consider. Following the distinction introduced by Vercelli [16] we consider a situation of "hard uncertainty" which contrasts with the classical situations of "soft uncertainty".

A situation of uncertainty is said to be soft whenever the available information concerning the value of the critical pollution threshold can be modelled by a unique (additive) probability measure. On the contrary, we talk about hard uncertainty when the available information is insufficient or incomplete to be modelled by such a unique fully reliable probability measure.

This general form of uncertainty seems to be particularly relevant to the discussion of many environmental risks. Indeed, concerning most of these questions the experts consider that the current state of scientific knowledge is not sufficient to measure precisely the incurred risk. They feel some kind of imprecision on the probability distribution of the catastrophic event itself.

In Vercelli [16], selected environmental problems involving hard uncertainties are discussed. It is essentially a matter of global threats such as the consequences of the ozon layer hole or the greenhouse effect.

The literature on imprecise probabilities proposes many mathematical models to cope with uncertainty situations that cannot be reduced to soft uncertainty (see Walley [17]). The point here is to implement the recent advances in imprecise probabilities theory to analytically solve a hard uncertainty situation in an optimal pollution control model.

The paper is organized as follows. In Section 2 we de-

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velop a model of optimal pollution control under hard uncertainty based upon the concept of closed random intervals. The model is then analytically solved and the optimal pollution accumulation path under hard uncertainty is compared with the solution obtained under soft uncertainty. Section 3 illustrates these results by numerical simulations of the model.

2 Optimal pollution control model

We consider an optimal pollution control model in an economy vulnerable to a catastrophic environmental event with irreversible effects. The instantaneous utility[1] of a representative individual $U(C_t)$ increases with the consumption level C_t but the production and consumption processes of this good give rise to pollution emissions εC_t, where ε is a constant pollution to consumption ratio. This pollution accumulates into a pollution stock S_t. The evolution law of S_t is given by:

$$\dot{S}_t = \varepsilon C_t - \alpha S_t \tag{1}$$

where α is the constant natural rate of pollution decay. Individuals are sensitive to pollution accumulation and the damage caused by pollution[2] $D(S_t)$ reduces their instantaneous welfare. The decision-maker (DM) has to trade-off consumption against pollution, given that consumption gives rise to pollution.

We only consider catastrophic events with so important consequences that it is always optimal for the economy to avoid them. This implies structural parameters of the economy such that the costs of avoiding the catastrophe are always less than the costs of the damages due to the catastrophe.

Assumption 1 *It is never optimal to let the catastrophe happen.*

In this model, a catastrophic event arises when the pollution stock exceeds a critical threshold X (which is an unknown but fixed environmental constant) that represents the maximum pollution level the ecosystem can tolerate. As usual X is modelled as a random variable (r.v.) and with c.d.f.

$$F_X(S) = P(X < S) \tag{2}$$

which is supposed to be continuously differentiable. $F_X(S)$ gives the probability that the castatrophe has already occurred for the pollution stock level S.

Let T be the (uncertain) date at which the catastrophe happens. As we shall see, under some assumptions T is also a r.v. whose distribution can be derived from that of X. The optimal pollution control problem can then be stated:

$$\begin{cases} \max\limits_{\{C_t\}} E_T \left\{ \int_0^T e^{-\rho t} \; (U(C_t) - D(S_t)) \; dt \right\} \\ \text{subject to} \quad \left| \begin{array}{l} \dot{S}_t = \varepsilon C_t - \alpha S_t \\ S_0 < X \text{ given} \end{array} \right. \end{cases} \tag{3}$$

The DM's problem is to maximize the intertemporal welfare of an infinitely living representative individual. Intertemporal welfare is the discounted sum of instantaneous welfare ($\rho > 0$ is the DM's discount rate). We suppose that once the catastrophe has occurred, the instantaneous welfare is reduced to zero for an infinite period of time ($U(C_t) - D(S_t) = 0$ for all $t \geq T$). The DM's objective is then the expected discounted flow of welfare, from date 0 to the (uncertain) date of the catastrophe, with respect to the probability distribution of T, given that at time zero the catastrophe has not happened yet. The notation $E_T\{\cdot\}$ denotes the expectation with respect to the distribution of T.

For simplicity we restrict our analysis to optimal consumption paths that correspond to monotonically increasing (or decreasing) pollution paths.

Along non-increasing pollution paths, we know for sure that the catastrophic event will never occur since for $S_0 < X$ and $S_t \leq S_0$ we have $P(X < S_t) = 0$ for all $t \geq 0$. The DM's objective then becomes:

$$\int_0^\infty e^{-\rho t} \; (U(C_t) - D(S_t)) \; dt \tag{4}$$

Along non-decreasing pollution paths it can easily be shown that the r.v. T is obtained from the r.v. X by an increasing transformation. The c.d.f. of T can then be deduced from that of X:

$$\begin{aligned} F_T(t) &\equiv P(T < t \,|T > 0) = P(X < S_t \,|X > S_0) \\ &= \frac{F_X(S_t) - F_X(S_0)}{1 - F_X(S_0)} \end{aligned} \tag{5}$$

We may also define the survival function $s_X(S_t)$ as the probability for the event not to have occurred yet at time t:

$$s_X(S_t) \equiv 1 - F_T(t) = \frac{1 - F_X(S_t)}{1 - F_X(S_0)} \tag{6}$$

The definition of the expected value implies the following equality :

$$E_T \left\{ \int_0^T e^{-\rho t} \; (U(C_t) - D(S_t)) \; dt \right\}$$

[1] We suppose a constant relative risk aversion (CRRA) utility function such that $U(C) \geq 0$, $U'(C) \geq 0$ and $U''(C) \leq 0$.

[2] We suppose that the marginal damage is non-decreasing in the pollution stock with $D(S) \geq 0$, $D'(S) \geq 0$ and $D''(S) \geq 0$. Under these assumptions, the welfare $U(C) - D(S)$ is concave in (C, S).

$$= \int_0^{\infty} \left(\int_0^t e^{-\rho t} \left(U\left(C_t\right) - D\left(S_t\right)\right) dt \right) dF_T\left(t\right)$$

Integrating this function by parts, given that $F_T(0) = 0$, $\lim_{t\to\infty} F_T(t) = 1$ and $1 - F_T(t) = s_X(S_t)$, leads to the following objective which is the expected value of intertemporal welfare.

$$\int_0^{\infty} s_X\left(S_t\right)\ e^{-\rho t}\ \left(U\left(C_t\right) - D\left(S_t\right)\right)\ dt \qquad (7)$$

Thus the solution to the problem (3) involves both the optimal pollution control problem without risk (for decreasing pollution paths) and the one under uncertainty studied on a restricted domain (i.e. non-decreasing pollution paths). This second problem is called an "auxiliary problem" by Tsur et Zemel [13].

When the distribution of X is precisely known, the auxiliary problem has a single solution (given the initial pollution stock level) and the optimal consumption and pollution paths can be fully determined. Here we suppose that the available information is incomplete or insufficient to completely specify the distribution of the critical pollution threshold X. More precisely, we suppose that the existence of a probability distribution for X can be hypothesized but, due to imprecise information, this distribution is only known to belong to some class of possible distributions.

2.1 Pollution accumulation without uncertainty

Knowing for sure the pollution critical threshold X beyond which the catastrophe occurs, the control problem may be written as:

$$\begin{cases} \max\limits_{\{C_t\}} & \displaystyle\int_0^{\infty} e^{-\rho t}\ \left(U\left(C_t\right) - D\left(S_t\right)\right)\ dt \\ \text{subject to} & \left| \begin{array}{l} \dot{S}_t = \varepsilon C_t - \alpha S_t \\ S_t \le X\ \forall t \ge 0 \\ S_0 \le X \text{ given} \end{array} \right. \end{cases} \qquad (8)$$

The optimality conditions are:

$$q = -U'\left(C\right)/\varepsilon \qquad (9)$$

$$\dot{q}/q = \rho + \alpha - \varepsilon\frac{D'\left(S\right) + \beta}{U'\left(C\right)} \qquad (10)$$

$$0 = \beta\left(X - S\right) \qquad (11)$$

where q_t is the co-state variable for the pollution stock and $\beta_t \ge 0$ is the Lagrange multiplier associated with the state constraint $S_t \le X$. The transversality condition is:

$$\lim_{t\to+\infty} e^{-\rho t} q_t S_t = 0 \qquad (12)$$

Totally differentiating with respect to time the first optimality condition (9) and writing the accumulation equation of the pollution stock (1), we obtain a dynamic system in C and S.

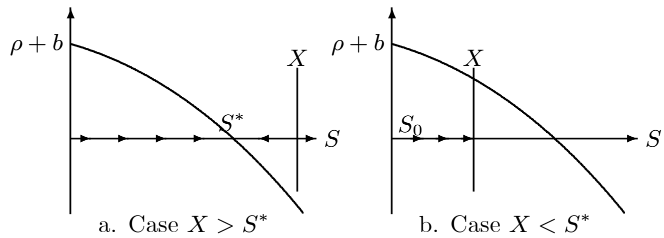
$$\begin{cases} \dot{C} = -\frac{C}{\sigma}\left(\rho + \alpha - \varepsilon\frac{D'(S)+\beta}{U'(C)}\right) \\ \dot{S} = \varepsilon C - \alpha S \end{cases} \qquad (13)$$

where $1/\sigma = -CU''/U'$ is the (constant) intertemporal elasticity of substitution for consumption and also the inverse of the relative risk aversion coefficient.

This system has a single interior stationary solution (C^*, S^*) such that $C^* = \alpha S^*/\varepsilon$ and S^* is the single solution to the equation:

$$\Phi\left(S\right) \equiv \rho + \alpha - \varepsilon\frac{D'\left(S\right)}{U'\left(\alpha S/\varepsilon\right)} = 0 \qquad (14)$$

where $\Phi(S)$ is strictly decreasing in S with $\Phi(0) > 0$ and $\lim_{S\to\infty}\Phi(S) < 0$. It can be shown that the steady-state (C^*, S^*) is a saddle-point equilibrium. This implies that if the pollution stock is above its steady-state value S^* the optimal path towards S^* is strictly non-increasing in S. On the contrary if $S_t < S^*$, the pollution stock should monotonically increase along the optimal path towards the steady-state. Figure 1 illustrates the pollution stock dynamics. Thus the optimal path of the certain case also



a. Case $X > S^*$ b. Case $X < S^*$

Figure 1: Pollution dynamics in the certain case

applies under uncertainty when the pollution stock is above its certain steady-state value S^* since this optimal path is then decreasing in S. On the other hand when $S_t < S^*$ the economy is vulnerable to the risk of ecological catastrophe and the optimal path is a solution to the auxiliary problem.

2.2 Auxiliary problems under hard uncertainty

Suppose that the available information concerning X is modelled as a (closed) random interval (c.r.i.) Γ defined on some fixed probability space $(\Omega, \mathcal{A}, P)$, i.e. Γ is a multi-valued mapping defined on Ω, which takes as values (non empty) closed intervals of the real numbers. Γ is supposed to be strongly measurable. The interpretation of this model is as follows. We suppose that the critical pollution threshold can be modelled as a real-valued r.v. X defined on $(\Omega, \mathcal{A}, P)$ but that we are facing difficulties in observing the outcome of the random experiment. More precisely we suppose that for each $\omega \in \Omega$ we cannot observe the exact outcome $X(\omega)$, but can only locate it in some non-empty closed interval $\Gamma(\omega)$, that is $X(\omega) \in \Gamma(\omega)$. This model, which takes both randomness and imprecision into account leads to a more general form of uncertainty than the classical one. To see this, we associate a class $\mathcal{X}$ of compatible real-valued r.v. with Γ

Definition 1 *A* compatible r.v. *(compatible with Γ) is a real-valued measurable mapping X defined on Ω, such that for all $\omega \in \Omega$, $X(\omega) \in \Gamma(\omega)$.*

When the available information is represented by a c.r.i., the "true" r.v. is not observable but one knows that it belongs to the class $\mathcal{X}$ of compatible r.v.'s. Each compatible r.v. induces a probability distribution and our information is completely described by a convex set of probability measures $\mathcal{P}$ whose members are the probability measures associated to the compatible variables. Moreover the lower enveloppe P_* of this set is a belief function and characterizes $\mathcal{P}$, i.e. $\mathcal{P}$ is the set of all probabilities that dominate P_*.

Then there exists as many auxiliary problems as random variables that are compatible with the available information, each of these problems being similar to an optimal pollution control problem under soft uncertainty. Let $F_X(S) = P(X < S)$ be the continuously differentiable c.d.f. of some compatible r.v. $X \in \mathcal{X}$. Moreover notice that only the non-decreasing paths along which the pollution stock remains below S^* are of interest for the auxiliary problem. An auxiliary problem can then be stated as:

$$\left\{ \begin{array}{ll} \max\limits_{\{C_t\}} \displaystyle\int_0^\infty s_X(S_t)\, e^{-\rho t} \left(U(C_t) - D(S_t)\right)\, dt \\ \text{subject to} & \left| \begin{array}{l} \dot{S}_t = \varepsilon C_t - \alpha S_t \\ 0 \leq S_0 \leq S_t \leq S^* \ \forall t \geq 0 \\ S_0 \leq S^* \text{ given} \end{array} \right. \end{array} \right. \tag{15}$$

where $s_X(S)$ is the survival function corresponding to X. The detailed resolution of this program is given in Chevé and Congar [3].

For each compatible r.v., problem (15) leads to a dynamic system in C and S.

$$\left\{ \begin{array}{l} \dot{C} = -\frac{C}{\sigma} \Big[\rho + \alpha - \varepsilon \frac{D'(S)}{U'(C)} \\ \quad + g_X(S)\left(\varepsilon C - \alpha S - U(C) + D(S)\right) - \frac{\delta - \gamma}{s_X(S)} \Big] \\ \dot{S} = \varepsilon C - \alpha S \end{array} \right. \tag{16}$$

where δ and γ are respectively the Lagrangian multipliers for the state constraints $S_t \leq S^*$ and $S_t \geq L$.

Interior stationary solutions to the system (16), denoted by $(\hat{C}_X, \hat{S}_X)$, are such that $\hat{C}_X = \alpha \hat{S}_X / \varepsilon$ and $\hat{S}_X$ is a solution to the following equation[3]:

$$\Pi_X(S) \equiv \Phi(S) - g_X(S)\, W(S) = 0 \tag{17}$$

where $W(S) = U(\alpha S/\varepsilon) - D(S)$ is the instantaneous welfare corresponding to a stationary pollution stock S, and

$$g_X(S) = -\frac{1}{s_X(S)} \frac{\partial s_X(S)}{\partial S} \tag{18}$$

is the "hazard rate"[4] that corresponds to the r.v. X with S the current pollution stock. $g_X(S)$ is the probability for the catastrophe to occur when the pollution stock marginally increases over S, given that it has not occurred yet.

$$g_X(S) = \lim_{\Delta \to 0} \frac{P(S \leq X < S + \Delta \mid X \geq S)}{\Delta} \geq 0 \tag{19}$$

Thus the optimal path under uncertainty at each time t only depends on the instantaneous catastrophe probability. This probability is summarized by the hazard rate $g_X(S)$. Moreover we notice that for all $W(S) \geq 0$, the steady-state pollution stock under uncertainty is necessarily less than the certain steady-state pollution stock, that is $\hat{S}_X \leq S^*$.

Since we have supposed that the catastrophe occurrence reduces the welfare to zero, Assumption 1 necessarily implies that the steady-state welfare in the certain case is positive, that is $W(S^*) \geq 0$. Thus soft uncertainty on the critical pollution threshold leads to a decrease in the steady state pollution stock. In other words, uncertainty leads to a more conservative behavior with respect to environmental quality (these results are detailed in Chevé [2]).

The resolution of the auxiliary problems associated with all the compatible variables enables us to work out all the possible pollution accumulation paths that are compatible with the available knowledge. More precisely we can find the optimal steady states for each compatible r.v. It can be shown that the set of all optimal steady states is an interval. The exact bounds of this interval can be computed when uncertainty is modelled by a c.r.i.

Using the formula (19) which defines the hazard rate for each r.v. $X \in \mathcal{X}$ as a conditional probability

[3] For convenience, we restrict our attention to the case where (17) has a single solution on $[0, S^*]$. This is the case for most common probability distributions and welfare functions.

[4] We abuse the notation since the hazard rate at time t should strictly be defined as the probability for the catastrophe to occur at the next time, given that it has not occurred yet. The "true" hazard rate is positively $\lambda_X(t) = g_X(S_t)\, \dot{S}_t$.

and the "full bayesian updating rule" (see for example Jaffray [9]), we associate a class of hazard rates $\mathcal{G} = \{g_X(S) : X \in \mathcal{X}\}$ with the class $\mathcal{X}$. The bounds of this class are defined as follows:

$$\begin{aligned} g^{**}(S) &= \sup\{g_X(S) : X \in \mathcal{X}\} \\ g_{**}(S) &= \inf\{g_X(S) : X \in \mathcal{X}\} \end{aligned} \tag{20}$$

These bounds can be computed using the upper and lower probabilities induced by the c.r.i.

Following Dempster [8], we consider the case of a c.r.i. defined by two real-valued r.v. defined on the same probability space. These random variables represent the bounds of the c.r.i. In this setting, the upper and lower probabilities of any interval may be simply expressed in terms of the joint c.d.f. of these two r.v. We then show that the lower bound of the hazard rates $g_{**}(S)$ is always equal to zero whatever the pollution stock level. It actually turns out that for each value of the pollution stock we can always find a compatible r.v. for which the catastrophe instantaneous probability is zero.

Note that for each pollution stock level the upper and lower hazard rates do not coincide with the hazard rates corresponding to the upper and lower r.v. that bound the class $\mathcal{X}$.

Using the upper and lower bounds of the hazard rate we define the following functions:

$$\begin{aligned} \Pi^{**}(S) &\equiv \Phi(S) - g^{**}(S)\,W(S) \\ \Pi_{**}(S) &\equiv \Phi(S) - g_{**}(S)\,W(S) = \Phi(S) \end{aligned} \tag{21}$$

Let $\hat{S}_{**}$ be the solution to the equation $\Pi^{**}(S) = 0$ and $\hat{S}^{**}$ be the solution to the equation $\Pi_{**}(S) = 0$, notice that since $g_{**}(S) = 0$ for each S, we have $\hat{S}^{**} = S^*$(notice the interchange of subscripts and superscripts in order to respect the ranking of the steady-states pollution levels). The imprecision on the probability distribution of the critical pollution threshold implies an interval of steady-state equilibria, each of them corresponding to a compatible r.v. Let $\mathcal{S}$ be the set of all compatible steady-states, that is $\mathcal{S} = \{\hat{S}_X : X \in \mathcal{X}\}$. The equilibrium $\hat{S}$ corresponding to the "true" r.v. X necessarily belongs to $\mathcal{S}$. Moreover, $\hat{S}_{**}$ and $\hat{S}^{**}$ are respectively the lower and upper bounds of the set $\mathcal{S}$.

2.3 Pollution accumulation dynamics

For each r.v. compatible with the available information the pollution accumulation dynamics is given by the resolution of the model under soft uncertainty. In this case, we can state the following propositions concerning the pollution stock dynamics (see Tsur and Zemel [13] and Chevé [2]).

Proposition 1 *When $\Phi(S) < 0$, the optimal path under uncertainty corresponds to the optimal path of the certain case. Along this path the pollution stock decreases.*
When $\Phi(S) > 0$, the optimal pollution path cannot be decreasing.
When $\Phi(S) > 0$ and $\Pi_X(S) < 0$, the optimal pollution path cannot be increasing.

This proposition implies that when $\Phi(S) \geq 0$ and $\Pi_X(S) \geq 0$, the optimal policy is to maintain the pollution stock at this level.

Proposition 2 *When $\Phi(S) > 0$ and $\Pi_X(S) > 0$, the optimal pollution path is necessarily increasing.*

When the probability distribution of X is known for sure and the equation $\Pi_X(S) = 0$ has a single solution on the interval $[0, S^*]$, the pollution accumulation dynamics is depicted in Figure 2. Finally, it can

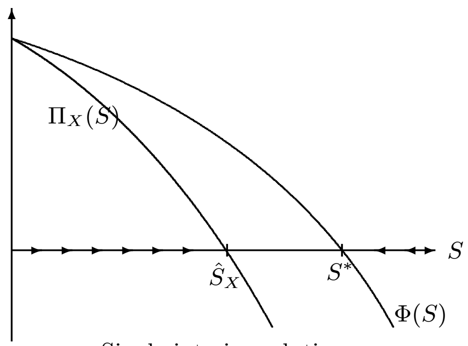


Figure 2: Pollution dynamics under soft uncertainty

be shown that whenever $\Pi_X(S)$ crosses the $S-$axis from above the steady-state is a saddle-point. On the contrary when $\Pi_X(S)$ crosses the $S-$axis from below it yields an unstable steady-state. Thus when the problem has a single solution, this equilibrium is necessarily a saddle-point.

Under hard uncertainty, that is when the "true" r.v. is only known to belong to the class $\mathcal{X}$ of compatible r.v., we obtain the Figure 3 (under the assumption that each equation (17) has a single solution on $[0, S^*]$).

By comparing the different pollution accumulation dynamics corresponding to each compatible r.v. we notice that these dynamics are compatible with each other on some part of the domain (the pollution stock evolves in the same direction for any compatible r.v.). We then deduce from this observation that:

- If $\Phi(S) \geq 0$ and $\Pi^{**}(S) \geq 0$ the pollution stock increases;

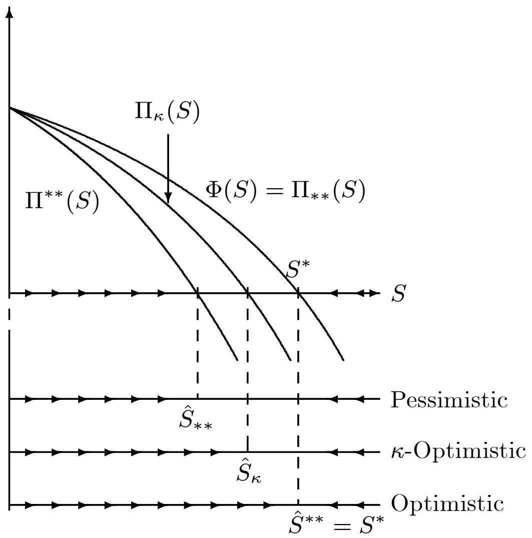


Figure 3: Pollution dynamics under hard uncertainty

- If $\Phi(S) \leq 0$ and $\Pi^{**}(S) \leq 0$ the pollution stock decreases towards S^*.

On the other hand if $\Pi^{**}(S) \leq 0$ and $\Phi(S) \geq 0$, that is if $S \in [\hat{S}_{**}, \hat{S}^{**}]$, the pollution stock dynamics cannot be solved. This is due to the irreducible imprecision concerning the distribution of X. In fact this dynamics depends on the DM's attitude towards the imprecision contained in the available information. Indeed he has to select a steady-state, in consumption and pollution, that belongs to the interval $[\hat{S}_{**}, \hat{S}^{**}]$. We may distinguish between three main different possible attitudes towards imprecision. The dynamics of the pollution stock is then the consequence of the particular choice of the DM.

The decision-maker is fully optimistic. We define a fully optimistic DM as a DM whose choice is based on the most optimistic available information. In the model such an individual decides on his steady-state consumption level by taking into account the smallest hazard rate which is compatible with the available information. Moreover it has been shown that whatever the pollution stock there always exists a compatible r.v. for which the hazard rate is zero, that is $g_{**}(S_t) = 0 \; \forall S_t$. Thus, when there exists some imprecision on the actual risk of catastrophe, a fully optimistic decision-maker acts as if there was no risk at all. The pollution accumulation dynamics is therefore the same as the one in the certain case. It means that when the pollution stock belongs to the interval $\mathcal{S}$, the pollution stock increases towards S^*.

The decision-maker is fully pessimistic. The opposite case is the one in which the DM takes his decisions considering the highest hazard rate that is compatible with the available information. In this case the optimal steady-state pollution stock $\hat{S}_{**}$ is the solution to the equation $\Pi^{**}(S) = 0$. If the pollution stock belongs to the interval $\mathcal{S}$ the optimal policy consists in maintaining the pollution stock at this level. Indeed it can be optimal for the pollution stock neither to decrease (with respect to the dynamics of the certain case since it corresponds to a decrease in welfare) nor to increase (with respect to the dynamics under uncertainty since the risk is too large compared with the increase in welfare).

Thus the imprecision leads to a more conservative behavior than simple randomness since $\hat{S}_{**} \leq \hat{S}_X$ $\forall X \in \mathcal{X}$.

The decision-maker is neither fully optimistic nor fully pessimistic. We can also consider the case of a DM who has a less extreme attitude towards imprecision than the preceding ones. Assume that an individual can be described by a certain "optimism index" constant over time, denoted $\kappa \in [0, 1]$. This DM select a steady-state by taking into account a hazard rate $g_\kappa(S_t)$ which is built as a convex combination of the extreme hazard rates.

$$g_\kappa(S) = \kappa g_{**}(S) + (1-\kappa)\, g^{**}(S)$$

The steady-state pollution stock is then $\hat{S}_\kappa$, the solution to the equation $\Pi_\kappa(S) = 0$, where $\Pi_\kappa(S) = \kappa \Pi_{**}(S) + (1-\kappa)\,\Pi^{**}(S)$ and $\hat{S}_\kappa \in \mathcal{S}$.

Such a behavior has a familiar form. It recalls the Hurwicz's criterion (Arrow et Hurwicz [1]) but applies to more general situations of uncertainty than the one of complete ignorance considered by Arrow and Hurwicz. Such representations of the behavior in partial ignorance situations described by a belief function on a finite set have been studied by Jaffray and Wakker [11]. The extension of this criterion to a dynamic framework is discussed in Jaffray [10]. The fully optimistic [resp. fully pessimistic] case corresponds to the maximum [resp. minimum] value of the optimism index, $\kappa = 1$ [resp. $\kappa = 0$].

Finally, we may suppose that a DM wishes the problem to come down to a situation of soft uncertainty by subjectively picking one random variable X in the class $\mathcal{X}$. Then his optimization problem corresponds to an auxiliary problem (and to the certain problem for $S_t \geq S^*$) and leads to a steady-state pollution stock $\hat{S}_X \in [\hat{S}_{**}, \hat{S}^{**}]$. If $S_t < \hat{S}_X$ the pollution path increases towards $\hat{S}_X$ and if $\hat{S}_X \leq S_t \leq S^*$ the pollution stock is maintained at this level. In this case, the underlying DM's attitude towards imprecision can be characterized in the Hurwicz's criterion framework. Indeed there exists a κ value such that $\hat{S}_\kappa$ coincides with the $\hat{S}_X$ associated with the particular X chosen by the DM.

3 A numerical example

In this last section we carry out numerical simulations of the optimal pollution accumulation steady-state under hard uncertainty. The first step is to build a random interval (the method based upon Dempster [8] is detailed in Congar [4]). We then compute the possible steady-state equilibria under hard uncertainty.

We choose some specific functional definitions for the utility of consumption and the damage caused by pollution:

$$U(C) = \frac{C^{1-1/\mu}}{1-1/\mu} \text{ and } D(S) = \phi \frac{S^{1+\theta}}{1+\theta} \tag{22}$$

The parameters values of the model have been arbitrarily set.

ϕ	θ	ε	α	ρ	μ
0.001	1	0.5	0.2	0.03	1.5

Under these assumptions the desired optimal pollution stock when there is no risk of catastrophe is $S^* = 57.13$. The flow of consumption is then equal to $C^* = 22.85$.

We choose the following parameters for the Gaussian joint-density function generating the random interval:

r	m_1	m_2	σ_1	σ_2
0.5	60	40	10	15

where r is the correlation coefficient. These values imply that the expected values of the extreme variables Y and Z are $\bar{Y} = 59.83$ and $\bar{Z} = 99.84$ which are both greater than the value of S^*. The expected value of the unobservable variable belongs to the interval $[59.83, 99.84]$.

We can compute the different steady-states pollution levels corresponding to the functions $\Pi^{**}(S)$, $\Pi_{**}(S) = \Phi(S)$, $\Pi_Y(S)$ and $\Pi_Z(S)$.

$\hat{S}_{**}$	$\hat{S}^{**}$	$\hat{S}_Y$	$\hat{S}_Z$
38.92	57.13	44.11	54.90

Each steady-state value of S compatible with the available information belongs to the interval $[\hat{S}_{**}, S^*] \simeq [38.92, 57.13]$. The steady-states corresponding to the extreme variables Y and Z also belong to this interval. However we notice *i)* that $\hat{S}_Y$ and $\hat{S}_Z$ are not the bounds of the interval $\mathcal{S}$ and *ii)* that $\hat{S}_Y$ is less than $\hat{S}_Z$. The second observation seems counter-intuitive but can be explained by the fact that the stochastic dominance relation between the two r.v. implies no relation of order between the hazard rates associated with these variables.

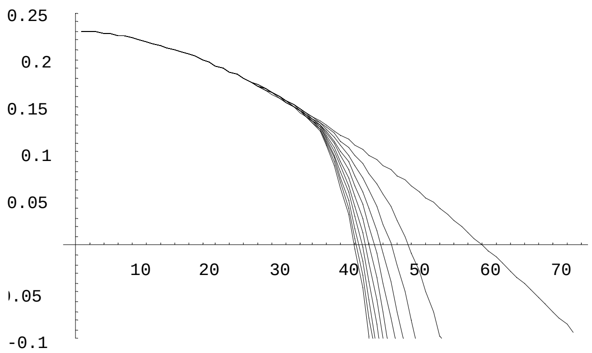


Figure 4: Steady-state equilibria and optimism index

The first result means that considering a situation of hard uncertainty yields some results that are not straightforward. Indeed these results are much different from the ones that would have been obtained considering a situation of soft uncertainty with a safety margin around the probability distribution of the critical pollution threshold.

Figure 4 and Table 1 represent a few steady-state pollution levels corresponding to the Hurwicz's criterion when the optimism index κ varies between zero and one.

κ	0	0.1	0.2	0.3	0.4	0.5
$\hat{S}_\kappa$	38.92	39.21	39.54	39.96	40.41	41.00
κ	0.6	0.7	0.8	0.9	1	
$\hat{S}_\kappa$	41.71	42.67	44.09	46.58	57.13	

Table 1: Steady-state pollution stock and optimism index

For the specific parameters values chosen in these numerical simulations, we notice that the equilibrium level of pollution does not evolve linearly with respect to the optimism index. Indeed even for a high optimism index ($\kappa = 0.9$) the steady-state pollution stock is closer to the fully pessimistic equilibrium than to the fully optimistic one. Thus, imprecision does not encourage the DM to neglect the risk of catastrophe except for the specific case of a fully optimistic DM ($\kappa = 1$).

According to these results it seems that the behavior of a fully pessimistic DM can be taken as an approximation of the behavior of the DM under hard uncertainty when there is no available information concerning the individuals' attitude towards imprecision. Such a recommendation is then in line with the precautionary principle. This principle in the Maastricht Treaty states that

"the absence of certainty given our current

> scientific knowledge should not delay the use of measures preventing a risk of large and irreversible damage to the environment at an acceptable cost".

According to this principle, we have shown that (hard) uncertainty on the risk of ecological catastrophe should lead to a more conservative policy than the one that should be implemented when only randomness prevails.

4 Concluding remarks

In this paper we have shown that the concept of hard uncertainty is tractable in applied models and especially in optimal pollution control models. Indeed closed random intervals provide a convenient framework to represent both theoretically and numerically situations of hard uncertainty in economic models.

The main result of this paper is that the imprecision on the risk of ecological catastrophe leads to a steady-state interval or to an interval of possible steady-states. The particular value to be chosen as a steady-state is left undecided unless the decision-maker's attitude towards imprecision is taken into account. The DM's ambiguity aversion (in the sense of aversion toward imprecision) is dealt with through the use of an Hurwicz's like criterion.

Computations of the model indicate that, without any information about the DM's attitude towards imprecision, a good approximation of his behavior could be the one of a fully-pessimistic DM. Moreover this fully-pessimistic DM chooses the lower bound of the interval of possible steady-states as an equilibrium. Thus a robust strategy under hard uncertainty is to choose the most conservative behavior. This result illustrates the precautionary principle.

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