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An Outline of a Comparative Foundation to Ambiguity Aversion

Paolo Ghirardato[§] **and Massimo Marinacci**[¶]

This outline contains a brief description of the main findings of a much longer work (Ghirardato and Marinacci [9]). In that paper we propose and characterize a formal definition of ambiguity aversion for a large class of preference models, which encompasses all the models which have been developed to include ambiguity in decision making. Using this notion, we then define and characterize ambiguity for the preferences which have a consistent ambiguity attitude.

The subjective expected utility (SEU) theory of decision making under uncertainty of Savage [15] is firmly established as the choice-theoretic underpinning of modern economic theory. However, such success has well known costs: SEU's simple and powerful representation is often violated by actual behavior, and it imposes some unwanted restrictions. In particular, Ellsberg [4]'s famous thought experiment convincingly shows that SEU cannot take into account the possibility that the information a decision maker (DM) has about some relevant uncertain event is vague or imprecise, and that such 'ambiguity' affects her behavior. In fact, Ellsberg observed that it affected his 'nonexperimental' subjects in a consistent fashion: Most of them preferred to bet on unambiguous rather than ambiguous events. Furthermore he observed that even when shown the inconsistency of their behavior with SEU, the subjects stood their ground "because it seems to them the sensible way to behave." This attitude has later been named *ambiguity aversion*.[1] Savage was well aware of this limit of SEU, for he wrote that

> [...] There seem to be some probability relations about which we feel relatively "sure" as compared with others. [...] The notion of "sure" and "unsure" introduced here is vague, and my complaint is precisely that neither the theory of personal probability, as it is developed in this book, nor any other device known to me renders the notion less vague. [15, pp. 57–58 of the 1972 edition]

Years after Ellsberg's provoking contribution, some extensions of SEU have been developed which allow ambiguity, and the DM's attitude towards it, to play a role in DMs' choices. In particular, two methods for extending SEU have established themselves as the standards of this literature. The first, originally proposed in Schmeidler [16], is to allow the DM's beliefs on the state space to be represented by non-additive probabilities, called *capacities*, and her preferences by *Choquet integrals* (which are just standard integrals when integrated with respect to additive probabilities). For this reason, this generalization is called the theory of *Choquet expected utility* (CEU) maximization. The second, developed by Gilboa and Schmeidler [10],[2] allows the DM's beliefs to be represented by multiple probabilities, and represents her preferences by the 'maximin' on the set of the expected utilities. This generalization is thus called the *maxmin expected utility* (MEU) theory. Some decision models based on these theories have recently received some attention

[§] Division of the Humanities and Social Sciences, California Institute of Technology, Pasadena, CA 91125, U.S.A.. E-mail: `paolo@hss.caltech.edu`.

[¶] Dipartimento di Scienze Economiche, Università di Bologna, Piazza Scaravilli 2, 40126 Bologna, Italy. E-mail: `marinacc@economia.unibo.it`.

[1] Other widespread names are 'uncertainty aversion' and 'aversion to Knightian uncertainty', both based on a well known distinction of 'risk' and 'uncertainty' attributed to Knight [12]. We think that 'uncertainty' should be reserved for defining *any* situation in which the results of the DM's possible actions are not known at the time of choice. On the other hand, 'Knightian uncertainty' is too cumbersome.

[2] Their paper contains the first axiomatic characterization, but the idea predates it. In fact, Savage was aware of it when writing the revision of [15], for he added to the paragraph quoted above a footnote saying that "[o]ne tempting representation of the unsure is to replace the person's single probability measure P by a set of such measures, especially a convex set."

by economists and political scientists interested in explaining phenomena at odds with SEU.[3]

In [9] we develop and employ a more general model of preferences with ambiguity attitudes, which has SEU, CEU, and MEU as special cases. The preferences described by the model, called *canonical preference relations*, are all those for which the ranking of consequences can be represented by a state-independent cardinal utility u, and the ranking of the bets on events by a unique numerical function ρ, that we call her *willingness to bet*. No restrictions beyond an obvious dominance condition are imposed on the ranking of more complex acts.

One of the reasons for the lasting success of SEU theory despite its limits is the elegant theory of the measurement of risk aversion, which has been developed starting from the seminal contributions of de Finetti [2], Arrow [1] and Pratt [14]. As Epstein [5] observes, no such theory is available for the notion of ambiguity aversion; a general theoretical foundation, which can be used for all the existing models of ambiguity averse behavior, is missing. The objective of this paper is to provide such a foundation: We propose a behavioral definition of ambiguity aversion for the most general decision-theoretic framework (Savage's), and we characterize it formally. Hopefully, this characterization will be useful for the literature of 'applications' of models of ambiguity aversion, as that of risk aversion was for the 'applications' of SEU. In particular, we expect it to be helpful in outlining the differences in the predictive scopes of traditional risk attitude and ambiguity attitude in specific situations.

To understand how we approach the problem, it is helpful to go back to the characterization of risk aversion in the SEU model. The following general approach to defining risk aversion was outlined by Yaari [17]. Given a state space S, let $\mathcal{F}$ denote a collection of 'acts', maps from S into $\mathbb{R}$ (e.g., monetary payoffs). We start by defining a *comparative* notion of risk aversion for SEU preferences: Say that $\succcurlyeq_2$ is *more risk averse than* $\succcurlyeq_1$ if the following implications hold for every 'riskless' (i.e., constant) act x and every 'risky' act f:

$$x \succcurlyeq_1 f \quad \Rightarrow \quad x \succcurlyeq_2 f \tag{1}$$

$$x \succ_1 f \quad \Rightarrow \quad x \succ_2 f \tag{2}$$

(where $\succ$ is the asymmetric component of $\succcurlyeq$). The two preferences are shown to induce identical beliefs, thus averting the confusion of different risk attitudes with different beliefs (cf. [17, p.317]). Having defined relative risk aversion, we next decide who to call a *risk neutral* DM. For instance, it can be an expected value maximizer. We then call *risk averse* a DM whose preference is more risk averse than that of a risk neutral DM. Interestingly, [17] shows that this definition has the usual concavity characterization.

This approach is based on two arbitrary choices. First, we established constant acts as riskless. Second, we established expected value maximization as the benchmark for defining risk aversion. Like the traditional 'internal' definition of risk aversion, this definition is fully *behavioral*, as it only uses the DM's preferences, but it is more general and powerful. In particular, it applies to a general subjective setting, rather than one with extraneous 'objective' probabilities. It also extends to non-SEU preferences.

We adopt Epstein [5]'s suggestion to follow a similar procedure to describe ambiguity attitude: We start from a comparative notion of 'more ambiguity averse than...', and then establish a benchmark; thus obtaining an 'absolute' definition of ambiguity aversion. As Savage, we use a very general framework with no extraneous devices. The only limitation we impose is a richness condition on the set of consequences.

Our development of the 'more ambiguity averse...' relation departs from the following intuitive considerations:

1. If a DM prefers an unambiguous act to an ambiguous one, a more ambiguity averse one will do the same.
2. If a DM prefers an ambiguous act to an unambiguous one, a less ambiguity averse one will do the same.

While this is very natural, the important question is of course: What do we mean by 'ambiguous' and 'unambiguous'? A tempting idea is to use the *weakest* preconceived notion of 'unambiguous' act: Say that an act is unambiguous if it is *constant*, and ambiguous otherwise. This is tantamount to saying that $\succcurlyeq_2$ is more ambiguity averse than $\succcurlyeq_1$ whenever Eqs. (1) and (2) hold. However, the following example shows that differences in risk attitude might intrude in the comparison, confusing the picture.

Example 1 Consider an (Ellsberg) urn containing balls of two colors: Black and Red. Two DMs are facing this urn, and they have no further information on its composition. The first DM has SEU preferences $\succcurlyeq_1$, with a utility function on the set of consequences $\mathbb{R}$ given by $u_1(x) = x$, and beliefs on the state space

[3] For instance, they have been applied to explaining the existence of incomplete contracts (Mukerji [13]), the existence of substantial volatility in stock markets (Epstein and Wang [6, 7], Hansen *et al.* [11]), or selective abstention in political elections (Ghirardato and Katz [8]).

of ball extractions $S = \{B, R\}$ given by

$$\rho_1(B) = \frac{1}{2} \quad \text{and} \quad \rho_1(R) = \frac{1}{2}.$$

The second DM also has SEU preferences, and identical beliefs: Her preference $\succcurlyeq_2$ is represented by $u_2(x) = \sqrt{x}$ and $\rho_2 = \rho_1$. Both (1) and (2) hold, but it is quite clear that this is due to differences in the DMs' risk attitudes, and not in their ambiguity attitudes: They both apparently disregard the ambiguity in their information. $\triangle$

We avoid this problem by developing a behavioral condition that guarantees that the two DMs have identical risk attitudes, without imposing any restriction on their ambiguity attitude. Our notion of comparative ambiguity aversion is therefore the conjunction of (1) and (2) with this behavioral condition. Throughout the paper, we narrowly use *risk attitude* to mean what explains a DM's choices among bets on the *same* event, differing only in the payoffs received for win or loss.[4] For canonical (in particular SEU, CEU and MEU) preference relations, this trait is fully characterized by the behavior of the utility function. Hence, if two DMs are ranked by comparative ambiguity they have the same utility function. This identity does *not* limit the scope of the absolute definition of ambiguity aversion. In fact, the latter is conceptually based on a comparison of the DM with a replica of herself, differing only in her preferences over acts which pay off on different events.

The second step in our exercise is choosing a benchmark against which to measure ambiguity aversion. We opt for what seems the natural candidate: SEU preferences. We thus call *ambiguity averse* a preference relation $\succcurlyeq$ for which there is a SEU preference 'less ambiguity averse than' $\succcurlyeq$. Ambiguity love and neutrality are then defined in the obvious way.

The first set of results of [9] deals with the characterization of these notions of ambiguity attitudes. The characterization of ambiguity neutrality holds for almost every preference relation and is simply stated: A preference is ambiguity neutral if and only if it has a SEU representation. That is, the *only* ambiguity neutral preferences in our sense are the SEU preferences. The results for ambiguity aversion and love hold for canonical preference relations. We show that a canonical preference relation is ambiguity averse (resp. loving) only if its willingness to bet is pointwise dominated by (resp. pointwise dominates) a probability.[5] This implies that all MEU preferences are ambiguity averse, as it is intuitive. In the CEU case, we show that the converse of the above statement is also true: Since then the willingness to bet coincides with the representing capacity, a CEU preference is ambiguity averse if and only if its capacity is dominated by some probability. That is, the 'core' of its capacity is non-empty. The characterization of relative ambiguity aversion for canonical preference relations follows immediately: If $\succcurlyeq_2$ is more ambiguity averse than $\succcurlyeq_1$ then $\rho_1 \geq \rho_2$. That is, a less ambiguity averse DM will express a uniformly higher willingness to bet. The converse is also true for CEU preferences, whereas for MEU preferences relative ambiguity is characterized by containment of the sets of probabilities.

An elementary application of this characterization is the following extension of a result of Dow and Werlang [3]. Suppose that a DM faces a world in which there are two assets: riskless money and a (possibly ambiguous) stock. Given that the DM has a budget of W units of money, how many units of the stock will she buy or sell short at a price p? It can be shown that if she has ambiguity averse CEU (or MEU) preferences, and her u is (weakly) concave, she may do neither for an interval of prices $[\underline{p}, \overline{p}]$ [3, Theorem 4.2]. It is immediate to verify that as her preferences become more ambiguity averse, the lowest (resp. highest) price at which she is willing to sell (resp. buy) will increase (resp. decrease), making the interval of 'inaction' larger. The converse is also true.

The second set of results of [9] deals with the characterization of ambiguity itself. An 'endogenous' notion of unambiguous act follows naturally from our earlier analysis: Say that an act is *unambiguous* if an ambiguity consistent (i.e., averse or loving) DM evaluates it in an ambiguity neutral fashion. The unambiguous *events* are defined to be those that unambiguous acts pay off on. We obtain the following simple characterization of the set of unambiguous events for canonical preference relations: For an ambiguity consistent DM with willingness to bet ρ, event A is unambiguous if and only if $\rho(A) + \rho(A^c) = 1$. In particular, this holds for CEU and MEU preferences, where ρ is given by the capacity and the lower envelope of the set of probabilities respectively. We also show that the sets of unambiguous acts and events satisfy some other interesting properties.

As an application of the previous analysis, we consider the classical Ellsberg problem with a 3-color urn. We show that the theory delivers the intuitive answers, once the information provided to the DM is correctly formalized into the problem. That is, if the fact that the extraction of a red ball (one of 30) is unambiguous is modelled according to the definition given above,

[4] In saying that all these are bets 'on' the event, we imply that a higher (resp. lower) payoff is always the result of the event obtaining (resp. not obtaining).

[5] The result we prove provides a *necessary* and sufficient condition (see [9, Theorem 7] for details).

then a CEU DM who is ambiguity averse in the sense described above will never choose in an *intuitively* ambiguity loving fashion, and under a natural assumption, he will moreover be intuitively ambiguity averse.

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