Applications of Possibility and Evidence Theory in Civil Engineering

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Abstract

This article is devoted to applications of *fuzzy set theory, possibility theory and evidence theory* in civil engineering, presenting current work of a group of researchers at the University of Innsbruck. We argue that these methods are well suited for analyzing and processing the parameter uncertainties arising in soil mechanics and construction management. We address two specific applications here: finite element computations in foundation engineering and a queueing model in earth work.

1 Introduction

There is increasing awareness in the engineering community that probability theory alone does not suffice for modelling the uncertainties arising in engineering problems. In view of the type of data commonly available, say in soil mechanics or construction management, far more flexible tools for assessing and processing subjective knowledge and expert estimates are needed.

Using risk analysis, it is usually easy for the planning engineer to provide focal sets for the fluctuations of the parameters involved at various risk levels. This opens the door for employing fuzzy sets, possibility theory or evidence theory. When these types of methods are effected for describing the input data, it is essential that arithmetical processing is possible in the engineering models (in finite elements, say) and results in output data of the same type. In fuzzy set theory, this is guaranteed by the extension principle (reducing the computations to evaluating the solution operators on the level sets), while in evidence theory, the computations can be done directly with the focal sets.

The purpose of this paper is to demonstrate two applications of these concepts in civil engineering: First, we show how finite element computations involving uncertain soil parameters can be performed and how uncertainties propagate through the engineering model, yielding robust assertions about the possible fluctuations of the output. The computational methods work equally well for data described by possiblity theory or by evidence theory. Second, we consider a queueing problem as typically arising in earth work at larger construction sites. We present an approach which employs a classical probabilistic queueing model with fuzzy input parameters, resulting in fuzzy state probabilities from which the required performance measures can be estimated; as the method of computation, we use fuzzy differential equations.

Generally speaking, possibility and evidence theory can aid the engineer in the planning phase as well as in control during construction. Engineering data are usually amenable to a rational description within this framework. Processing the information through the engineering models provides a robust basis for specification and risk assessment.

Apart from the two exemplifying situations presented here, we refer to our papers [8, 9, 10, 17] for further applications. Among the rapidly increasing civil engineering literature on applications of fuzzy sets, we refer to [11, 19] as typical examples and recommend the recent expository volume [1] and the references therein for an overview.

2 Finite element method with vague and uncertain parameters

2.1 Preliminaries

We begin by introducing some preliminary definitions, c.f. [7, 30]. Let $X \subseteq \mathbb{R}^k$ be a nonempty set, $F = \{F_1, \dots, F_n\}$ a finite set of distinct subsets (focal sets) of X and the set function $m : F \longrightarrow [0,1]$ a basic probability assignment on F. Then

$$\underline{P}(A) = \mathrm{Bel}(A) = \sum_{\substack{F_i \subseteq A \\ F_i \in F}} m(F_i) \tag{1}$$

is the belief measure on $(X, \wp(X))$ or the lower probability of A and

$$\overline{P}(A) = \mathrm{Pl}(A) = \sum_{\substack{F_i \cap A \neq \varnothing \\ F_i \in F}} m(F_i) \tag{2}$$

the plausibility measure on $(X, \wp(X))$ or the upper probability of A.

Let $f : X \subseteq \mathbb{R}^k \longrightarrow \mathbb{R} : (x_1, \dots, x_k) \mapsto f(x_1, \dots, x_k)$ be a continous function, $G = \{G_1, \dots, G_n\}$ with $G_i = f(F_i)$ the set of the images of the focal sets and $A \subseteq f(X) \subseteq \mathbb{R}$. Then

$$\mathrm{Bel}_f(A) = \mathrm{Bel}(f^{-1}(A)) = \sum_{\substack{F_i \subseteq f^{-1}(A) \\ F_i \in F}} m(F_i) \tag{3}$$

$$= \sum_{\substack{f(F_i) \subseteq A \\ f(F_i) \in G}} m(f(F_i)) = \sum_{\substack{G_i \subseteq A \\ G_i \in G}} m(G_i) \tag{4}$$

where $m(G_i) = m(f(F_i)) := m(F_i)$ and in the same way

$$\mathrm{Pl}_f(A) = \sum_{\substack{G_i \cap A \neq \varnothing \\ G_i \in G}} m(G_i). \tag{5}$$

If we have instead a fuzzy subset $M \subseteq \mathbb{R}^k$ with membership function $\mu_M : \mathbb{R}^k \longrightarrow [0,1]$ then $\mu_{f(M)}$ is obtained by Zadeh's extension principle or, since f is continous, built up by the images of the α-level sets $[M]_\alpha$ of M: $[f(M)]_\alpha = f([M]_\alpha)$, $\alpha \in (0,1]$. Recall that the α-level set is given by $[M]_\alpha = \{x \in X : \mu_M(x) \geq \alpha\}$.

2.2 The finite element method

Let $B \subseteq \mathbb{R}^3$ be a homogeneous isotropic body, which is subjected to certain loads. We want to compute the displacements $u(x) = (u_1(x), u_2(x), u_3(x))^T$ and stresses $\sigma_{ij}(x)$ at points $x = (x_1, x_2, x_3)^T$ in B.

At first we need some preliminary definitions. Let $\sigma = \sigma^T$ be the stress tensor and ε the strain tensor which are defined by

$$\sigma_{ij} = E\left(\frac{1}{1+\nu}\varepsilon_{ij} + \frac{\nu\delta_{ij}}{(1-2\nu)(1+\nu)}\sum_{k=1}^{3}\varepsilon_{kk}\right) \tag{6}$$

and

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{7}$$

for $i,j = 1, \dots, 3$ where $E > 0$ is the elastic modulus and $\nu \in [0, \frac{1}{2})$ Poisson's ratio. E and ν are two parameters describing the material properties of the body B.

For each spatial direction $i = 1, \dots, 3$ we split ∂B, the surface of B, into sets Γ_{1i} and Γ_{2i} with $\Gamma_{1i} \cap \Gamma_{2i} = \varnothing$ and $\Gamma_{1i} \cup \Gamma_{2i} = \partial B$. Further let

$$\|v\|_E^2 = \int_B \sum_{i,j=1}^{3} \sigma_{ij}(v)\varepsilon_{ij}(v)\mathrm{d}x \tag{8}$$

be the elastic energy in the body B and S the Sobolev space defined by

$$S = \left\{v : \|v\|_E < \infty,\ v_i|_{\Gamma_{1i}} = 0, i = 1, \dots, 3\right\}. \tag{9}$$

The displacement $u \in S$ on B is the solution of the following system of partial differential equations:

$$\begin{gathered} \mathrm{div}\sigma(u) + f = 0 \\ u_i = 0 \text{ on } \Gamma_{1i}, \quad i = 1, \dots, 3 \\ (\sigma(u) \cdot n)_i = g_i \text{ on } \Gamma_{2i}, \quad i = 1, \dots, 3, \end{gathered} \tag{D}$$

where $n = n(x)$ is the normal of ∂B in x, $f : B \longrightarrow \mathbb{R}^3$ the body load and where $g_i : \Gamma_{2i} \longrightarrow \mathbb{R}$ are the surface tractions.

Remark: Let u^1 be the solution of (D) for $E = 1$ and fixed ν. Then $u = \frac{1}{E}u^1$ is the solution of (D) for arbitrary $E > 0$ and the same ν as above. Further, it holds that $\sigma = \sigma^1$ where σ is obtained from u and σ^1 from u^1. These results can be proven simply by inserting $\frac{1}{E}u^1$ into (D).

The weak formulation of (D) which we need for the finite element method is obtained by multiplying the differential equation by $v \in S$ and integration by parts:

Find $u \in S$ such that

$$a(u,v) = F(v) \text{ for all } v \in S \tag{W}$$

where

$$a(u,v) = \int_B \sum_{i,j=1}^{3} \sigma_{ij}(v)\varepsilon_{ij}(v)\mathrm{d}x \tag{10}$$

$$F(v) = \int_B fv\mathrm{d}x + \sum_{i=1}^{3}\int_{\Gamma_{2i}} gv\mathrm{d}x. \tag{11}$$

The standard method in civil engineering for obtaining an approximate solution u^h of the above problem is to use the finite element method where the infinite dimensional Sobolev space S is replaced by finite subspace

$$S^h = \{v \in S : v_i|_K \in P_n(K) \text{ for all } K\} \tag{12}$$

with basis $\{\phi_1, \ldots, \phi_N\}$. The sets K are the so called finite elements into which B is partitioned, e.g. tetrahedra or hexahedra for 3-dimensional problems or triangles or quadrilaterals for 2-dimensional problems. On such an element K the functions $v_i(x) = \sum_{k=1}^{N} c_{ik}\phi_k(x)$, $i = 1, 2, 3$, are polynomials of degree n. Then (W) becomes:

Find $u_i^h = \sum_{k=1}^{N} c_{ik}\phi_k(x) \in S^h$ such that for all $v_i^h = \sum_{l=1}^{N} d_{il}\phi_{il}(x) \in S^h$, $i = 1, 2, 3$, it holds that

$$a(u^h, v^h) = F(v^h). \tag{13}$$

This leads to a system of linear equations for the coefficients c and therefore to u^h. For more details see [12, 26].

2.3 Numerical method for finite elements with vague parameters

We want to stress that the finite element computations are done for *fixed* parameters $(E, \nu) \in (0, \infty) \times [0, \frac{1}{2})$ or, in general, for fixed parameters $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Lambda \subseteq \mathbb{R}^k$.

We will now simply view the results of a finite element computation as functions

$$s_{[\lambda_1, \ldots, \lambda_k]} = s_{[\lambda]} : B \longrightarrow \mathbb{R} \tag{14}$$

where

- s represents a component u_i^h of the displacements (the displacements in x_i-direction) or an element σ_{ij}^h of the stress tensor which we obtain from the solution u^h;
- the index $[\lambda_1, \ldots, \lambda_k]$ means that the finite element computation is done for the fixed k parameters $\lambda_1, \ldots, \lambda_k$ describing the properties of the body $B \subseteq \mathbb{R}^3$.

In general $s_{[\lambda]}$ can also represent the solutions of more complicated problems, e.g. elasto-plastic material properties with additional parameters.

If we consider the results as depending on the parameters $(\lambda_1, \ldots, \lambda_k) \in \Lambda$ we get a function

$$s : \Lambda \longrightarrow S^h : \lambda \mapsto s_{[\lambda]}. \tag{15}$$

Here we are interested in what happens in single points $x \in B$ depending on $(\lambda_1, \ldots, \lambda_k) \in \Lambda$. So we introduce the function

$$s_{(x_1, x_2, x_3)} = s_{(x)} : \Lambda \longrightarrow \mathbb{R} : \lambda \mapsto s_{[\lambda]}(x) \tag{16}$$

which evaluates s at a fixed point x of B. The function $s_{(x)}$ is continous if Λ is a set of feasible parameter values. So $s_{(x)}$ is a continous function f as considered in the definitions of Pl_f and Bel_f.

Let $A \subseteq \mathbb{R}$ and $F = \{F_1, \ldots, F_n\} \subseteq \wp(\Lambda)$ with a given probability assignment m. For computing $\mathrm{Pl}_{s_{(x)}}(A)$ and $\mathrm{Bel}_{s_{(x)}}(A)$ for given points $x \in B$ we have to obtain the images $G_i = s_{(x)}(F_i)$ of the focals F_i, that is to determine the extremal values of $s_{(x)}$ on F_i. We approximate the function s by an interpolant $\overline{s}$ on $D = \bigcup_{i=1}^{n} F_i$. Thus D is discretized and $s_{[\lambda]}$ is computed on nodes $\lambda \in D$. This can be done by a finite element program package. Then the optimization on F_i $(i = 1 \ldots, n)$ getting the images is performed with $\overline{s}_{(x)}$.

For this process we have to use as much information as possible about $s_{(x)}$ (e.g. monotonicity) and the shape of the F_i to reduce the dimension of the optimization problem (see the example below).

In case the parameters are modelled by fuzzy sets, our computations can be equally performed with the α-level sets in place of the focal sets. Other methods have been proposed in the literature: enclosing the α-level sets of the solution by means of interval arithmetic [20]; an approach nonequivalent to the extension principle is in [28, 29].

2.4 Numerical example

We have applied our methods to raft foundations [10] and recently to finite element computations in tunnelling. Here, for the purpose of illustration, we use a less complex example. Fig. 1 depicts a profile of a linear elastic soil medium. This soil medium is subjected to a load of 200 kN/m resulting from a foundation in x_3-direction.

This problem can be reduced to a two-dimensional problem (plane-strain) for which we can use the same formulas as above but with indices up to 2 only. As boundary conditions the horizontal displacements on the left and right side and the vertical displacements on the bottom line have to be zero. The elastic constants E and ν are assumed to be given by four two-dimensional focal sets F_1, F_2, F_3 and F_4 (see Fig. 2) with probability masses $m(F_1) = 0.2$, $m(F_2) = 0.3$, $m(F_3) = 0.3$ and $m(F_4) = 0.2$.

For the design of the foundation it is important to estimate the expected displacement at surface level. Thus a typical question of engineering interest could be, for example, wether the displacement u_2 at point $(0, 40)$ is less than -0.2 m. The results for our example are (with $s = u_2^h$):

$$\overline{P}_{s_{(0,40)}}((-\infty, -0.2]) = \mathrm{Pl}_{s_{(0,40)}}((-\infty, -0.2]) = 0.8$$

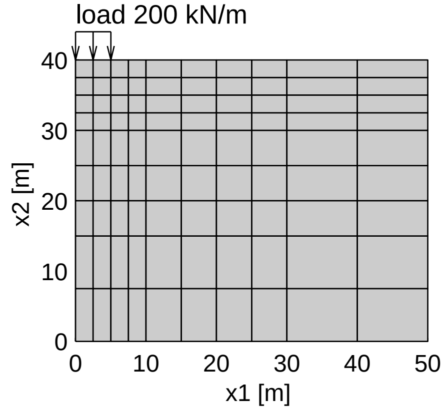

Figure 1: Finite element mesh of soil profile.

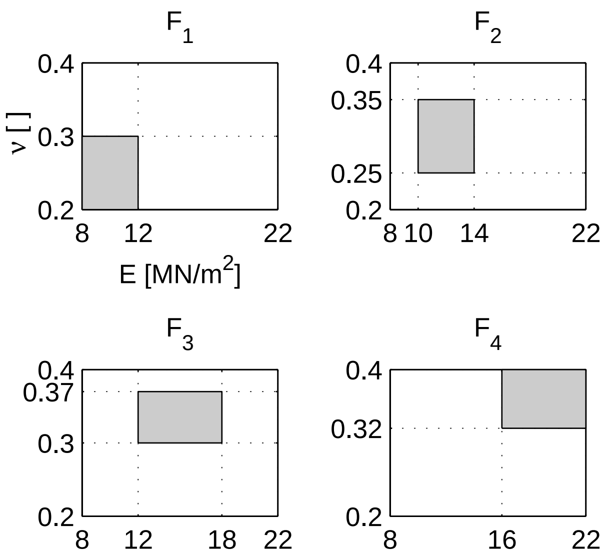

Figure 2: Focal sets F_1, F_2, F_3 and F_4.

and

$$\underline{P}_{s_{(0,40)}}((-\infty,-0.2]) = \mathrm{Bel}_{s_{(0,40)}}((-\infty,-0.2]) = 0.5$$

and the following images G_i of the focals F_i:

$G_1 = [-0.407, -0.25]$ m, $G_2 = [-0.315, -0.201]$ m,
$G_3 = [-0.25, -0.152]$ m, $G_4 = [-0.183, -0.118]$ m.

The computation is done as follows: Using the remark in Section 2.2 about the dependence of the solution on E and that the F_i are Cartesian products of intervals we have here a one-dimensional set $D = \{1\} \times [0.2, 0.4]$ on which we have to compute the interpolants $\overline{s}$. Therefore we compute solutions $s_{[1,\nu]}$ for e.g. $\nu \in \{0.2, 0.225, 0.25, \ldots, 0.375, 0.4\}$. Then we obtain the interpolant $\overline{s}$ by e.g. piecewise linear interpolation.

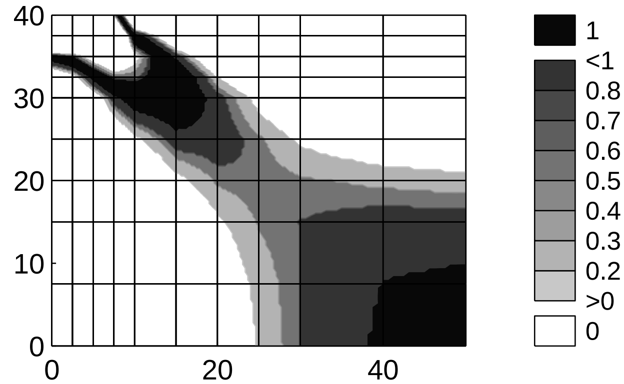

Figure 3: $\overline{P}_{\sigma_{11}^h}([-25,-15]) = \mathrm{Pl}_{\sigma_{11}^h}([-25,-15])$.

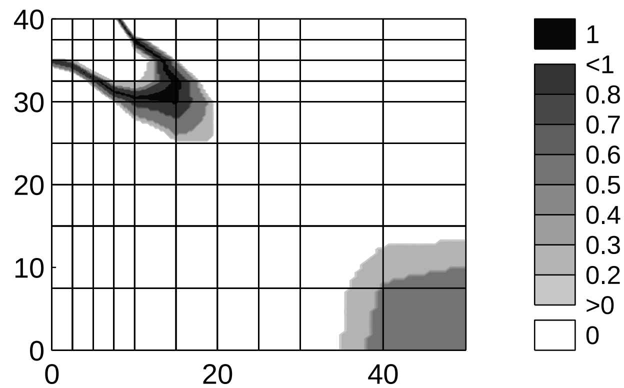

Figure 4: $\underline{P}_{\sigma_{11}^h}([-25,-15]) = \mathrm{Bel}_{\sigma_{11}^h}([-25,-15])$.

We write the F_i as Cartesian products $F_i = E_i \times \nu_i$ of intervals E_i and ν_i. We obtain the intervals $G_i = [G_i^{\text{left}}, G_i^{\text{right}}]$ for the results in point x as follows:

Let $\overline{s}$ be the interpolant on D for a displacement u_i^h. Then we get by interval division (see for example [21]):

$$G_i = \frac{[\min_{\nu \in \nu_i}\{\overline{s}_{(x)}(\nu)\}, \max_{\nu \in \nu_i}\{\overline{s}_{(x)}(\nu)]}{E_i}. \qquad (17)$$

Further, let $\overline{s}$ be the interpolant on D for a component σ_{ij} of the stress. Then we get:

$$G_i = [\min_{\nu \in \nu_i}\{\overline{s}_{(x)}(\nu)\}, \max_{\nu \in \nu_i}\{\overline{s}_{(x)}(\nu\}]. \qquad (18)$$

2.5 Visualizing the results

The results $s_{[\lambda]}$ of deterministic finite element computations, like stresses σ_{ij}^h, are often visualized by plotting areas $C = \{(x_1, x_2) : a_1 \le s_{[\lambda]}(x_1, x_2) \le a_2\}$ in different colors for different intervals $A = [a_1, a_2]$ on cross sections of B.

Extending this visualizing concept we plot $\overline{P}_{s_{(x_1,x_2)}}(A)$ or $\underline{P}_{s_{(x_1,x_2)}}(A)$ for each point (x_1, x_2). In Fig. 3 and 4 the stress σ_{11}^h (obtained from the above example) is depicted for the interval [-25,-15] kN/m^2. The shaded areas correspond to specific values of upper and lower probability of the event $\{\sigma_{11}^h \in [-25, -15]\}$ at (x_1, x_2).

In the case of using fuzzy sets instead of fuzzy measures, the set C defined above is a fuzzy set; regions of equal membership degree can be visualized in a similar manner.

3 Queueing models in earth work

This application concerns a typical queueing problem in civil engineering, as arising in earth work at larger construction sites. A purely possibilistic approach to queueing models can be found in [3]. Our method is in the spirit of [18]: we will develop a probabilistic model with fuzzy parameters, namely, in the terminology of [18], an (FM/FM/1):N queueing system. In contrast to the Markov chain methods of [18], our approach is based on the differential equations for the probabilities $p_k(t)$ that k customers are present in the system at time t and serves as an application of our previous work [22] on fuzzy differential equations.

3.1 The civil engineering problem

We consider a closed loop queueing system, consisting of a single server (excavator) and N customers (transport vehicles). After loading by the server, the vehicles transport and unload the material and return to the server. Due to variations in the service and return times, a waiting queue will build up in front of the server. Input parameters are the average service time $1/\mu$ (service rate μ) and the average return time $1/\nu$ (driving rate ν) of each vehicle. The engineering problem is to design the system in the most cost efficient way. Having chosen a certain excavator, this will chiefly be decided by the number of transport vehicles employed: Too few vehicles will incure costs due to idle time of the server, while too many will incure costs due to waiting time spent in the queue. The transportation and excavation costs per unit time given, the essential performance index is the avarage time T needed by each transport vehicle to complete a full run. Given $1/\nu$, this in turn is determined by the arrival rate λ and $T = N/\lambda$, N the total number of vehicles.

In the project planning phase, the designing engineer has to determine the input parameters of the system $1/\mu, 1/\nu$ in order to calculate the required capacity of the equipment. The service rate μ of the excavator depends on a large number of uncontrollable conditions: soil parameters, like grain structure, angle of internal friction, loosening; accessibility of construction site; effective slewing angle of the excavator; meteorological conditions, and so on. Available data just are not amenable to statistical methods. However, as noted in the introduction, the planning engineer can usually provide lower and upper bounds at various risk levels, using his experience and extrapolating data of former projects. In conclusion, the input parameters μ and ν can be rationally described by focal sets. We assume that the information is consonant; then we may assemble these focal sets into fuzzy numbers, following e. g. the procedure outlined in [7]. The main goal is to calculate the possibility distribution of the total run time T, from which other indicators of engineering interest can be computed.

We shall base our analysis on a standard probabilistic queueing model, which assumes that the service time and the return time are exponentially distributed with expectation values $1/\mu, 1/\nu$, leading to the Markov $(M/M/1) : N$ queueing system (see e. g. [16]). In this case, the decisive state variables are the probabilities $p_k(t), k = 0, \cdots, N$ that k customers are present in the queueing system at time t. These probabilities give preliminary information on the initial behavior of the system, but will chiefly be used to compute the stationary state with limiting probabilities $\pi_k = \lim_{t\to\infty} p_k(t)$. The stationary state will serve as a good approximation to the behavior of the system (simulations with actual data from construction management indicate that it is usually reached within one to two hours). Given fuzzy data $\tilde{\mu}, \tilde{\nu}$, all these probabilities are fuzzy as well. As shown below, the fuzzy run time $\tilde{T}$ can be computed from there without difficulty.

3.2 The crisp queueing model

Following standard arguments (see e.g. [16]) one can deduce the system

$$\begin{aligned} p_0'(t) &= -N\nu p_0(t) + \mu p_1(t)\,, \\ p_k'(t) &= (N-k+1)\nu p_{k-1}(t) \\ &\quad -(\mu + (N-k)\nu)p_k(t) + \mu p_{k+1}(t)\,, \\ &\quad k = 1, \cdots, N-1\,, \qquad (19) \\ p_N'(t) &= \nu p_{N-1}(t) - \mu p_N(t) \end{aligned}$$

with constraint

$$\sum_{k=0}^{N} p_k(t) = 1\,. \qquad (20)$$

We normally will assume "deterministic" initial data $p_j(0) = 1$ for some j, $p_i(0) = 0$ for $i \neq j$. We note that $(\sum p_k(t))' = 0$ so that the constraint (20) is automatically satisfied for all times iff satisfied initially. There is a unique equilibrium state given by

$$\pi_0 = \left(\sum_{n=0}^{N} \frac{N!}{(N-n)!}\left(\frac{\nu}{\mu}\right)^n\right)^{-1} \quad (21)$$

$$\pi_k = \pi_0 \frac{N!}{(N-n)!}\left(\frac{\nu}{\mu}\right)^n \quad k = 1, \ldots, N. \quad (22)$$

Proposition. The probabilities $p_k(t), k = 0, \ldots, N$ converge to the equilibrium probabilities π_k as $t \to \infty$, uniformly when μ and ν vary in compact subsets of $(0, \infty)$.

Proof: We are dealing with a system of ordinary differential equations of the form

$$p'(t) = A(\mu, \nu)p(t)$$

with constraint (20). Considering the columns of the matrix $A(\mu, \nu)$, Gershgorin's theorem immediately shows that the eigenvalues $\kappa_j, j = 0, \ldots, N$ of $A(\mu, \nu)$ are zero or have real part strictly less than zero. One of the eigenvalues equals zero, say $\kappa_0 = 0$, giving the stationary state $A(\mu, \nu)\pi = 0$. We will show that the algebraic multiplicity of κ_0 equals 1. First, the geometric multiplicity is 1 (otherwise, this would contradict the uniqueness of the equilibrium state $A(\mu, \nu)\pi = 0, \sum \pi_k = 1$). Suppose the algebraic multiplicity is greater than 1. Then the Jordan form of A would contain a nontrivial Jordan block with diagonal elements equal to 0. For a suitable choice of initial data, this would imply that there is a direction in which the solution $\exp(tA)p(0)$ grows at least linearly as $t \to \infty$ while remaining bounded in the other directions. This contradicts the constraint (20). It follows that $A(\mu, \nu)$ has one eigenvalue $\kappa_0 = 0$ with multiplicity 1, while the other eigenvalues have real part strictly negative. Therefore, $\exp(tA)p(0)$ converges to the equilibrium state π uniformly in μ, ν as $t \to \infty$ under the constraint (20).

Remark: The convergence of $p_k(t)$ to an equilibrium state follows from probabilistic arguments as well [16]. However, we shall need the uniform convergence asserted in the proposition.

By standard arguments from queueing theory, the average number of vehicles in the queue (in the equilibrium state) is easily computed as $L = N - (\mu/\nu)(1 - \pi_0)$. By equating the arrival with the departure rate we get $\lambda = \nu(N - L) = \mu(1 - \pi_0)$. Finally, we obtain $T = N/\lambda$ for the average total run time of each vehicle.

3.3 Fuzzy differential equations

For our purpose, it suffices to consider a linear system of the form

$$x'(t) = A(\gamma)x(t) \quad (23)$$

where A is an $(n \times n)$-matrix depending smoothly on a parameters $\gamma \in \mathbb{R}^m$ and $x(t) \in \mathbb{R}^n$. We fix (crisp) initial data and denote the value of the solution at time t by $x(t) = S_t(\gamma)$. When some of the components of γ are fuzzy, we can view it as a fuzzy subset $\tilde{\gamma}$ of $\mathbb{R}^m$ (from now on, we use the tilde notation to distinguish a fuzzy variable from its realizations). Our approach will be to apply the Zadeh extension principle [31] to the (continuous) map $\gamma \to S_t(\gamma) : \mathbb{R}^m \to \mathbb{R}^n$, and we will consider $S_t(\tilde{\gamma})$ as the *fuzzy solution* at time t. In this way, the α-level sets of $S_t(\tilde{\gamma})$ are precisely the images of the α-level sets of $\tilde{\gamma}$ under the solution operator S_t. We prefer this approach because

- viewing the α-level sets of $\tilde{\gamma}$ as focal sets, the output consists of the focal sets induced by solving the system of differential equations. This is precisely the information of engineering interest, namely the possible fluctuations of the output at level α, given the fluctuations of the input;
- one can rewrite (23) as a prolongued system, putting all fuzzy parameters in the initial data, and then apply the extension principle to the equation and to the time evaluation map at time t. This gives a fuzzy solution concept of (23), for which $t \to S_t(\tilde{\gamma})$ is the unique solution (see [22]).

Our approach is equivalent to the one using the flow in [2] and in [23]. We remark that other - nonequivalent - approaches have been undertaken: imbedding fuzzy sets into metric spaces [5, 13, 14, 25], differentiation of bounding curves of α-level sets [13, 15, 27], parametrized fuzzy numbers [24]; see also [4, 6, 13] for a study of the interrelations. For our numerical computations, we use the algorithm developed in [22] which gives the components $S_t^k(\tilde{\gamma})$ of the fuzzy solution $S_t(\tilde{\gamma})$. The Cartesian product of the fuzzy components yields the smallest non-interactive fuzzy vector containing the fuzzy solution.

We end this section by introducing some notions needed below. By a *fuzzy number* $\tilde{r}$ we mean a fuzzy subset of $\mathbb{R}$ such that all level sets $[\tilde{r}]_\alpha$ as well as its support are compact intervals and $[\tilde{r}]_1$ consists of a single point. We use the distance (see e. g. [5]) between fuzzy numbers $\tilde{r}, \tilde{s}$,

$$d(\tilde{r}, \tilde{s}) = \sup_{\alpha \in (0,1]} d_H([\tilde{r}]_\alpha, [\tilde{s}]_\alpha)$$

where d_H denotes the Hausdorff distance.

3.4 The fuzzy queueing system

As noted in Section 3.1, we suppose that the input parameters μ, ν are fuzzy numbers, which we denote by $\tilde{\mu}, \tilde{\nu}$, following our notational convention. We apply the extension principle to the solution operator $(\mu, \nu) \to S_t(\mu, \nu)$ of system (19) with constraint (20). According to Section 3.3, this produces the fuzzy solution

$$\tilde{p}(t) = S_t(\tilde{\mu}, \tilde{\nu})$$

which is dominated by the non-interactive vector with components

$$\tilde{p}_k(t) = S_t^k(\tilde{\mu}, \tilde{\nu}), \tag{24}$$

each describing the fuzzy (marginal) probability that k vehicles are in the system at time t. We note that the fuzzy equilibrium probabilities $\tilde{\pi}_k$ could be computed from formula (22) and (21) with the aid of the extension principle. However, having our machinery at hand, it is simpler to pass to the limit in the expressions (24) for the time-dependent fuzzy marginals as $t \to \infty$. The convergence is guaranteed by the following result:

Proposition. The fuzzy probabilities $\tilde{p}_k(t)$ converge to $\tilde{\pi}_k$ in the sense that

$$d(\tilde{p}_k(t), \tilde{\pi}_k) \to 0 \text{ as } t \to \infty\,.$$

Proof: By the Proposition in Section 3.2, $p_k(t) = S_t^k(\mu, \nu) \to \pi_k(\mu, \nu)$ uniformly as μ, ν vary in the supports of $\tilde{\mu}, \tilde{\nu}$. Therefore, the Hausdorff distance of the respective α-level sets converges to zero uniformly in $\alpha \in (0, 1]$ as well, as required.

What concerns the application in civil engineering, an important construction management task is to lay out the capacity of the system in the planning phase and allocate the required equipment. The basic performance parameter in this case is the total time needed by each vehicle on average to complete a full run. As noted in Section 3.2, it is obtained from $\tilde{\pi}_0$ as the fuzzy number

$$\tilde{T} = N/\tilde{\mu}(1 - \tilde{\pi}_0);$$

due to the fact that $\tilde{\mu}$ and $\tilde{\pi}_0$ are interactive, this is actually an upper estimate. Various further information can be extracted from there. First, the total cost per transported unit mass is proportional to $\tilde{T}$ and $C_V + C_S/N$, where C_V and C_S denote the cost per unit time of a single vehicle and the server, respectively. Thus estimates (with their degrees of possibility) of how the total cost emanates depending on the number N of vehicles can be obtained.

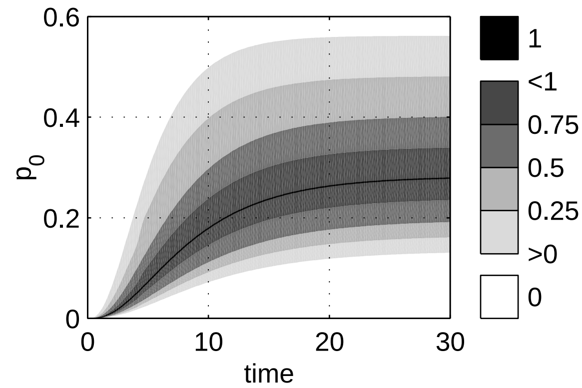

Figure 5: α-level sets of $\tilde{p}_0(t)$.

Second, the average number of completed runs by all vehicles together in a given period D is given by $DN/\tilde{T} = D\tilde{\mu}(1 - \tilde{\pi}_0)$. Thus the degree of possibility of achieving a certain performance in a given period of time can be computed as well, giving the basis for assessing the risk of not achieving a required threshhold.

Below we present a computational example with three transport vehicles ($N = 3$) and fuzzy mean serving time given by a triangular fuzzy number with supporting interval $[2, 6]$ and center at $z = 4$, while the average return time is modelled as a triangular fuzzy number with support $[9, 11]$ and center at $z = 10$. The initial state was taken deterministic as described above with $p_3(0) = 1$ (all three transport vehicles present at start). Fig. 5 shows the α-level sets of the fuzzy time-dependent probability $\tilde{p}_0(t)$ for $0 \le t \le 30$. One can read off that at time $t = 30$ the equilibrium state is almost reached; the probability $\tilde{\pi}_0$ can be approximated by a triangular fuzzy number with supporting interval $[0.13, 0.56]$ and center at 0.28.

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