Totally Monotone Core and Products of Monotone Measures

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Abstract

Several approaches to the product of non-additive monotone measures (or capacities) are discussed and a new approach is proposed. It starts with the Möbius product [2] of totally monotone measures and extends it by means of a supremum to general monotone measures. The sup runs over sets of totally monotone measures. These sets are defined like the core of monotone measures (or cooperative games). The new product is compatible with the partial order for arbitrary monotone measures.

Keywords. Monotone measure, product measures, core of cooperative game.

1 Introduction

The Copenhagen group Hendon et al. [2] has shown convincingly, that among the different approaches for the product of non-additive monotone measures (or capacities) the version using the Möbius representation has best properties if one restricts to totally monotone measures. Especially it extends the classical product of additive measures. But in general this *Möbius product*, performed with monotone measures is not monotone any more.

Koshevoy [3] proposes a class of so called *triangulation products*. First he defines the product of $\{0,1\}$-valued monotone measures as a sup of totally monotone $\{0,1\}$-valued measures (or unanimity games) and extends this definition by piecewise linearity with respect to a triangulation of the set of normed monotone measures, the wedges being $\{0,1\}$-valued monotone measures. Constructing a triangulation containing the totally monotone measures as a simplex, he can overcome the abovementioned shortcoming of the Möbius product. Another (canonical) one among the triangulation products is the chain product. It is again a monotone measure, but it is not additive if both factors are additve measures.

We propose a variant of the core, the *totally monotone core* of a set function (or cooperative game) where totally monotone measures (or belief functions) replace the additive ones in the very definition. Taking the sup of the products of the respective core elements, we extend the Möbius product for totally monotone measures to arbitrary measures. This new product of monotone measures is - like the triangulation products - always monotone and the main achievement is that it behaves monotone in both factors.

Compatibility of the product with the partial order of monotone measures is essential if one is concerned with imprecise probabilities, e.g. for the issue of generalised stochastic independence. Since imprecision of probabilities can be described by a pair of conjugate monotone measures, one would also appreciate to have a product which is compatible with conjugation of monotone measures. With respect to conjugation we get only the partial result (9). Another open question concerns associativity of the product if the factors are not totally monotone or $\{0,1\}$-valued.

2 Chain and Möbius representation of a monotone measure

For simplicity we suppose that Ω is a nonvoid finite set. Throughout, a (normed) **set function** ν on the power set 2^Ω is a real valued function $\nu : 2^\Omega \to [0,1]$ with $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$. A set function ν is called a **monotone measure** if $A \subset B$ implies $\nu(A) \le \nu(B)$. A set function ν is called k-**monotone**, $k \ge 2$, if for $A_1, ..., A_k \subset \Omega$

$$\nu(\bigcup_{i=1}^{k} A_i) + \sum_{\substack{I\subset\{1,\dots,k\}\\ I\neq\emptyset}} (-1)^{|I|}\ \nu(\bigcap_{i\in I} A_i) \ge 0.$$

2-monotonicity is also called **supermodularity** or **convexity**. **Submodularity** is the corresponding property with the reversed inequality sign. ν is **totally monotone** or a **belief function** if it is monotone and k-monotone for any $k \geq 2$.

The familiar method of representing a real function on a finite set as linear combination of indicator functions will be applied to a monotone measure ν. First, ν has only finitely many distinct upper level sets $\{\emptyset\} \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_n \subset 2^\Omega$, i.e. $\mathcal{S}_i = \{A \subset \Omega \mid \nu(A) \geq t_i\}$ for some t_i and $1 = t_1 > t_2 > \cdots > t_n > 0$. Defining η_i by $\eta_i(A) = 1$ if $A \in \mathcal{S}_i$ and $= 0$ else, we get a representation of ν as convex combination of $\{0,1\}$-valued monotone measures,

$$\begin{aligned} \nu &= \sum_{i=1}^{n} c_i\,\eta_i \quad \textit{with distinct } \{0,1\} - \textit{valued} \\ & \quad \eta_1 \leq \cdots \leq \eta_n \textit{ and all } c_i > 0\,. \end{aligned} \tag{1}$$

Furthermore, any monotone measure ν has only one representation of type (1). We call it the **chain representation** of ν.

There are still simpler $\{0,1\}$-valued monotone measures than the η above, the 'unanimity games' of cooperative game theory. A **unanimity game** u_K for 'coalition' $K \in 2^\Omega$, $K \neq \emptyset$, is the monotone measure defined by $u_K(A) = 1$ iff $A \supset K$ and $= 0$ else. Every $\{0,1\}$-valued monotone measure η can be written as a maximum (we use the sign $\vee$) of unanimity games

$$\begin{aligned} \eta &= \bigvee_j u_{K_j} \quad \textit{where } K_j \textit{ are} \\ & \quad \textit{the minimal sets } K \textit{ with } \eta(K) = 1\,. \end{aligned} \tag{2}$$

η is also a linear combination of unanimity games with integer coefficients, which can easily be derived from (2) applying the principle of inclusion exclusion (see [1]). Combining this result with (1) we get a representation of an arbitrary monotone ν as linear combination of unanimity games, the coefficient of u_K being denoted $\mu^\nu(K)$,

$$\nu = \sum_{\substack{K \subset \Omega \\ K \neq \emptyset}} \mu^\nu(K)\; u_K\,. \tag{3}$$

Again, this representation is unique since the unanimity games are linearly independent. This is the well known **Möbius representation** of ν. Compared to the chain representation, it has the disadvantage that the coefficients $\mu^\nu(K)$ may assume negative values and the advantage that the $\{0,1\}$-valued monotone measures used in the representation, the unanimity games, are totally monotone, whereas generally the η_i in (1) don't have this property. If the coefficients in (3) are all non-negative, $\mu^\nu \geq 0$, then ν is totally monotone like the u_K, and the converse is also true.

Finally, in this preparatory section, we recall the definition of the **conjugate** $\overline{\nu}$ of a set function ν,

$$\overline{\nu}(A) := 1 - \nu(\Omega \setminus A)\,.$$

In applications to imprecise probabilities the pair $(\nu, \overline{\nu})$ describes the **imprecision** or **uncertainty**, especially if $\nu \leq \overline{\nu}$. The difference $\overline{\nu} - \nu$ is called **ambiguity** or **vagueness** (see e.g. [1] Section 9).

3 The totally monotone core

Combining (1) with (2), it can easily be seen that ν is a maximum of linear combinations of unanimity games with positive coefficients, hence a maximum of totally monotone measures. This nice property had been the motive to introduce the following variant of the (additive) core.

The **additive core**[1] of a set function ν on 2^Ω is

$$\begin{aligned} \mathrm{core}_+\nu &:= \{\alpha \mid \alpha \text{ additive on } 2^\Omega,\, \alpha \leq \nu, \\ & \qquad \alpha(\Omega) = \nu(\Omega)\} \end{aligned}$$

It is well known that $\mathrm{core}_+\nu \neq \emptyset$ if ν is submodular and monotone. Analogously we define the **totally monotone core**[2] as

$$\begin{aligned} \mathrm{core}_\ll\nu &:= \{\beta \mid \beta \text{ totally monotone on } 2^\Omega,\, \beta \leq \nu, \\ & \qquad \beta(\Omega) = \nu(\Omega)\}. \end{aligned}$$

Obviously $\mathrm{core}_+\nu \subset \mathrm{core}_\ll\nu$ and, as noted already,

$$\mathrm{core}_\ll\nu \neq \emptyset\,, \quad \nu = \bigvee_{\beta \in \mathrm{core}_\ll \nu} \beta \tag{4}$$

for arbitrary monotone ν. More general we have

Proposition 3.1 *Given a monotone measure ν on 2^Ω and a chain $\mathcal{K} \subset 2^\Omega$, there exists $\beta \in core_\ll\nu$ such that $\beta|\mathcal{K} = \nu|\mathcal{K}$.*

Proof Let $\emptyset = K_0 \subset \ldots \subset K_n = \Omega$ be the chain $\mathcal{K}$. We construct β via its Möbius representation,

$$\begin{aligned} \mu^\beta(K_i) &:= \nu(K_i) - \nu(K_{i-1}) \geq 0,\, i = 1, \ldots, n, \\ \mu^\beta(A) &:= 0 \quad \text{for } A \in 2^\Omega \setminus \mathcal{K}\,. \end{aligned}$$

Then $\beta(K_i) = \sum_{L \subset K_i} \mu^\beta(L) = \nu(K_i)$ for all i. For arbitrary $A \subset \Omega$ select the maximal index i such that $K_i \subset A$. Then $\beta(A) = \beta(K_i) = \nu(K_i) \leq \nu(A)$, which completes the proof. □

As a corollary we get that for any function X on Ω there exists a $\beta_X \in \mathrm{core}_\ll\nu$ with $\int X d\nu = \int X d\beta_X$.

[1] In cooperative game theory the usual notation is core $\nu := \{\alpha \mid \alpha \text{ additive on } 2^\Omega,\, \nu \leq \alpha,\, \alpha(\Omega) = \nu(\Omega)\} = \mathrm{core}_+\overline{\nu}$.

[2] $\mathrm{core}_\ll\nu$ might also be an interesting solution concept in cooperative game theory. $\beta \in \mathrm{core}_\ll\nu$ defines a distribution $\mu^\beta \geq 0$ of total wealth $\nu(\Omega)$ to all coalitions $K \neq \emptyset$, not only to singletons like the elements of $\mathrm{core}_+\nu$. Notice that with the present notations $\nu(K)$ is the *maximal* value, coalition K (together with its subcoalitions) can get.

Here the integral is the Choquet integral. Again, for the additive core the corresponding result holds only if ν is submodular.

Example 3.1 $\mathrm{core}_+ u_K = \emptyset$ if K has at least two elements. But $\mathrm{core}_\ll u_K$ consists of all convex combinations of the unanimity games u_L, $L \supset K$, $\mathrm{core}_\ll u_K = \{\sum_{L\supset K} a_L u_L \mid a_L \geq 0 \text{ for } L \supset K,\ \sum_{L\supset K} a_L = 1\}$. Since $\mathrm{core}_+ \overline{u_K} \neq \emptyset$ we get also $\mathrm{core}_\ll \overline{u_K} \neq \emptyset$ □

4 The Möbius product

Let Ω_1 and Ω_2 be finite sets and $\Omega := \Omega_1 \times \Omega_2$ their cartesian product. The problems with the product arise from the fact that $2^{\Omega_1} \times 2^{\Omega_2}$ does not identify via $(A_1, A_2) \mapsto A_1 \times A_2$ with $2^{\Omega_1 \times \Omega_2}$ but only with a proper subfamily. The minimal requirement for a product $\otimes$ of two monotone measures ν_1 on 2^{Ω_1} and ν_2 on 2^{Ω_2} is

$$\nu_1 \otimes \nu_2(A_1 \times A_2) = \nu_1(A_1)\,\nu_2(A_2)\,, \quad A_i \subset \Omega_i\,. \qquad (5)$$

But how to extend it for arbitrary sets in Ω?

For unanimity games (5) implies

$$u_{K_1} \otimes u_{K_2} = u_{K_1 \times K_2}\,, \quad K_i \subset \Omega_i\,. \qquad (6)$$

Hendon et al. [2] advocate to extend this formula by bilinearity, applied to the Möbius representation (3) of the monotone measures ν_1 and ν_2,

$$\nu_1 \otimes_\mu \nu_2 := \sum_{K_1, K_2} \mu^{\nu_1}(K_1)\mu^{\nu_2}(K_2)\, u_{K_1} \otimes u_{K_2}\,, \qquad (7)$$

where K_i runs through all nonempty subsets of Ω_i, $i = 1, 2$. We will call the product $\nu_1 \otimes_\mu \nu_2$ the **Möbius product**, whence the index μ in the notation. The right hand side of (7) simultaneously gives the Möbius representation of $\nu_1 \otimes_\mu \nu_2$. Notice that the Möbius coefficients $\mu^{\nu_1 \otimes \nu_2}(K)$ of the product are vanishing if K is not a product set $K_1 \times K_2$.

Proposition 4.1 *(i) The Möbius product is linear in both factors and (5) holds;*

(ii) if ν_1, ν_2 are additive then $\nu_1 \otimes_\mu \nu_2$ is the classical additive product;

(iii) $\nu_1 \otimes_\mu \nu_2$ is totally monotone iff ν_1, ν_2 are totally monotone (i.e. $\mu^{\nu_i} \geq 0$);

(iv) if ν_2 is totally monotone then $\lambda_1 \leq \nu_1$ implies $\lambda_1 \otimes_\mu \nu_2 \leq \nu_1 \otimes_\mu \nu_2$;

(v) $(\nu_1 \otimes_\mu \nu_2) \otimes_\mu \nu_3 = \nu_1 \otimes_\mu (\nu_2 \otimes_\mu \nu_3)$.

If one restricts to totally monotone set functions, property (v) implies that the Möbius product is monotone in both factors.

Proof The proofs are straightforward, only (iv) needs some explanation. We may assume $\nu_2 = u_{K_2}$ for some $K_2 \subset \Omega_2$. For $A \subset \Omega_1 \times \Omega_2$ define $A_1 := \bigcup_{K_1 \times K_2 \subset A} K_1 \subset \Omega_1$ and show $\lambda_1 \otimes_\mu u_{K_2}(A) = \lambda_1(A_1)$. Since the same equality holds with ν_1, the assertion follows from $\lambda_1 \leq \nu_1$. □

Fubini's Theorem, for the Choquet integral, holds in the following sense ([1]).

Proposition 4.2 *If ν_1 is totally monotone, then*

$$\int X d\nu_1 \otimes_\mu \nu_2 \quad \leq \quad \int\int X(\omega_1, \omega_2) d\nu_1(\omega_1) d\nu_2(\omega_2)$$

and equality holds if ν_2 is additive.

A serious shortcoming of the Möbius product is, that monotonicity is not inherited by the product. Thus, it provides a satisfactory product only for totally monotone measures (see (iii)).

Example 4.1 Let $\eta_i(A_i) = 1$ iff $A_i \neq \emptyset$, $A_i \subset \Omega_i$, i.e. $\eta_i = \overline{u_{\Omega_i}}$. Suppose $\Omega_i = \{a_i, b_i\}$, $i = 1, 2$, and let $\eta := \eta_1 \otimes_\mu \eta_2$, then $\eta(\{(a_1, a_2), (b_1, b_2)\}) = 2$ whereas $\eta(\{(a_1, a_2), (a_1, b_2), (b_1, b_2)\}) = 1$, whence η is not monotone. □

5 Triangulation products

To circumvent this unpleasant phenomenon Koshevoy [3] proposes further products in representing normed monotone measures as convex combinations of $\{0, 1\}$-valued monotone measures. Since these representations are not unique, he fixes a triangulation of the set of all monotone measures with the $\{0, 1\}$-valued monotone measures as wedges, with respect to which there is a unique representation.

First the product of $\{0, 1\}$-valued monotone measures η_1, η_2 has to be defined. According to (2) and (6) it is natural to set ([3])

$$\begin{aligned} \eta_1 \otimes \eta_2 \quad &:= \bigvee_{j_1, j_2} u_{K_{1,j_1}} \otimes u_{K_{2,j_2}} \\ &= \bigvee_{\substack{\eta_i(K_i)=1 \\ i=1,2}} u_{K_1} \otimes u_{K_2} \end{aligned} \qquad (8)$$

where we use the representations (2) $\eta_i = \bigvee_{j_i} u_{K_{i,j_i}}$ on 2^{Ω_i}, $i = 1, 2$. The product $\eta_1 \otimes \eta_2$ is a $\{0, 1\}$-valued monotone measure on $2^{\Omega_1 \times \Omega_2}$ and it behaves monotone and associative for $\{0, 1\}$-valued monotone measures.

In Example 4.1 we get as desired $\eta_1 \otimes \eta_2(A) = 1$ iff $A \neq \emptyset$, i.e. $\overline{u_{\Omega_1}} \otimes \overline{u_{\Omega_2}} = \overline{u_{\Omega_1 \times \Omega_2}}$. In other words, the product (8) of the non-informative pairs $(u_{\Omega_i}, \overline{u_{\Omega_i}})$, $i = 1, 2$, now is the non-informative pair $(u_\Omega, \overline{u_\Omega})$ on the product set $\Omega = \Omega_1 \times \Omega_2$. This result generalises to arbitrary unanimity games,

$$\overline{u_{K_1}} \otimes \overline{u_{K_2}} = \overline{u_{K_1 \times K_2}} = \overline{u_{K_1} \otimes u_{K_2}}\,. \tag{9}$$

The easy proof relies on the general fomula $\overline{u_K} = \bigvee_{\omega \in K} u_{\{\omega\}}$.

Koshevoy constructs triangulations which contain the set of totally monotone measures as a simplex. Given such a triangulation, every monotone measure belongs to exactly one simplex of minimal dimension and can uniquely be represented as a convex combination of the extreme points (i.e. $\{0,1\}$-valued monotone measures) of that simplex. Then the product (8) is extended by bilinearity like the Möbius product above. But now, since only linear combinations with positive coefficients are used, the product is always a monotone measure.

There is another interesting triangulation where a simplex is defined by a chain of $\{0,1\}$-valued monotone measures, being their convex hull. Since this triangulation is canonical there is no need to refer explicitly to triangulations for defining the corresponding product. The **chain product** ([3], [1]) of monotone measures applies the chain representations $\nu_i = \sum_{k_i} c_{i,k_i} \eta_{i,k_i}$, with distinct $\eta_{i,1} \le \cdots \le \eta_{i,n_i}$ and all $c_{i,k_i} > 0$, $i = 1, 2$, (see (1)),

$$\nu_1 \otimes_\kappa \nu_2 := \sum_{k_1,k_2} c_{1,k_1} c_{2,k_2}\ \eta_{1,k_1} \otimes \eta_{2,k_2}\,. \tag{10}$$

Since the c_{i,k_i} are positive, $\nu_1 \otimes_\kappa \nu_2$ is a monotone measure, too, and (5) holds. The $\{0,1\}$-valued monotone measures on the right hand side of (10) are not totally ordered, so this is not the chain representation of the product $\nu_1 \otimes_\kappa \nu_2$. Hence associativity is not obvious like for the Möbius product, it still is an open problem. Another shortcoming here is, that for additive monotone ν_1, ν_2 it happens that the chain product is not additive.

6 An alternative product

Here we generalise formula (8) directly. Notice that among the $\{0,1\}$-valued monotone measures the totally monotone ones are the unanimity games, which appear on the right hand side of that formula. Analogously to (2), monotone measures ν_i on 2^{Ω_i} can be represented as (4)

$$\nu_i = \bigvee_{\beta_i \in \mathrm{core}_\ll \nu_i} \beta_i, \qquad i = 1, 2.$$

Now we define our new **product**,

$$\nu_1 \otimes \nu_2 := \bigvee_{\substack{\beta_i \in \mathrm{core}_\ll \nu_i \\ i=1,2}} \beta_1 \otimes_\mu \beta_2\,.$$

It inherits the good properties, which the Möbius product has for totally monotone measures.

Proposition 6.1 *(i) $\nu_1 \otimes \nu_2$ is a monotone measure for arbitrary monotone measures ν_1, ν_2 and (5) holds;*

(ii) $\nu_1 \otimes \nu_2$ coincides with the Möbius product $\nu_1 \otimes_\mu \nu_2$ for totally monotone measures ν_1, ν_2;

(iii) $\nu_1 \otimes \nu_2$ coincides with (8) for $\{0,1\}$-valued monotone measures ν_1, ν_2;

(iv) $\lambda_i \le \nu_i$, $i = 1, 2$, implies $\lambda_1 \otimes \lambda_2 \le \nu_1 \otimes \nu_2$;

(v) associativity

$$\nu_1 \otimes (\nu_2 \otimes \nu_3) \quad = \quad (\nu_1 \otimes \nu_2) \otimes \nu_3$$

holds at least if all ν_i are totally monotone or $\{0,1\}$-valued.

Proof The proofs of (i),(ii),(iv) and (v) are straightforward, but notice that monotonicity of the Möbius product ((v) in Proposition 4.1) plays an important rôle, not only for (iv), but also for (ii).

For (iii) with $\{0,1\}$-valued $\nu_i = \eta_i$ we have to show

$$\bigvee_{\substack{\beta_i \in \mathrm{core}_\ll \eta_i \\ i=1,2}} \beta_1 \otimes_\mu \beta_2 \quad = \quad \bigvee_{\substack{\eta_i(K_i)=1 \\ i=1,2}} u_{K_1} \otimes u_{K_2} \tag{11}$$

For the K_i on the right hand side $u_{K_i} \in \mathrm{core}_\ll \eta_i$ so that $\ge$ holds in (11). If in equation (7) for $\beta_1 \otimes_\mu \beta_2$ the product of unanimity games $u_{K_1} \otimes u_{K_2}$ appears with a coefficient $\neq 0$, then $\eta_i(K_i) = 1$, $i = 1, 2$, so that $u_{K_1} \otimes u_{K_2}$ appears also on the right hand side of (11). Since the (nonnegative) coefficients (7) sum to 1 we get $\le$ in (11). □

7 Conclusion and outlook

We have combined ideas of existing models for the product of monotone measures to construct a product which is again monotone and is compatible with the partial order in both factors. Also a dual approach is feasable. It starts with the conjugate $\overline{u_K}$ of the unanimity games, represents an arbitrary monotone measure as linear combination of the $\overline{u_K}$ (dual Möbius representation) reverses inequalities and replaces the sup $\vee$ with the inf $\wedge$. The resulting dual product equals $\overline{\overline{\nu_1} \otimes \overline{\nu_2}}$ and things would be easy if this equalled $\nu_1 \otimes \nu_2$ (in (9) we got this equality only for unanimity games). Otherwise the question arises, which pair should be taken as the product of imprecision pairs $\nu_i \le \overline{\nu_i}, i = 1, 2$? There are four possibilities $\nu_1 \otimes \nu_2 \le \overline{\nu_1} \otimes \overline{\nu_2}$, $\nu_1 \otimes \nu_2 \le \overline{\overline{\nu_1} \otimes \overline{\nu_2}}$, $\overline{\overline{\nu_1} \otimes \overline{\nu_2}} \le \overline{\nu_1} \otimes \overline{\nu_2}$ and $\overline{\overline{\nu_1} \otimes \overline{\nu_2}} \le \overline{\nu_1 \otimes \nu_2}$.

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